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MATHEMATICS MAGAZINE

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OUR CONTRIBUTORS

Harold F. Mathis, Associate Professor of Electrical Engineering, University of Oklahoma, and a member of the staff of the University of Oklahoma Research Institute, was born in Texas in 1916. He is a graduate of the University of Oklahoma (B.S. '39) and Texas A. & M. College (M.S. '41). A member of Sigma Xi, of the A.M.S. and of five engineering societies, Mr. Mathis also serves on the Committee on Basic Sciences and its Subcommittee on Applied Mathematics of the American Institute of Electrical Engineers. He does research principally on antennas and microwaves. His main mathematical interest is in new applications of mathematics to electrical engineering.

Edwin F. Beckenbach, Professor of Mathematics, U.C.L.A., was born in Dallas, Texas in 1906. A graduate of the Rice Institute (A.B. '28, A.M. '29, Ph.D. '31), he was a National Research Fellow in Mathematics at Princeton, Ohio State and Chicago Universities, 1931-33, and then returned to Rice as an instructor. He also served on the faculties of the Universities of Michigan and Texas before his appointment to the University of California, Los Angeles in 1945. Professor Beckenbach is an editor of the new Pacific Journal of Mathematics, and has also served as associate editor of the American Mathematical Monthly. His mathematical interests include complex variable theory, convex, harmonic and subharmonic functions and differential geometry, and he has contributed to all of these disciplines.

John R. Lowe, 32 years old, was born in Louisville, Kentucky. He studied accounting and mathematics in night schools and by correspondence and did accounting work until 1941. He then became interested in the use of I.B.M. punched card equipment for accounting and decided to specialize in that field. While at Lockheed Aircraft Corporation during the war, Mr. Lowe and some others tried doing some of the engineering computation with punched-card accounting machines as an experiment, at that time quite a new idea. The experiment was so successful that he has been doing this work ever since. In 1947 he was employed by Douglas Aircraft Co., Inc., to organize a machine computing group which has been quite successful. He has published several papers on computing subjects.

Nilos Sakellariou, Professor, University of Athens, was born in Hagios Petros, Arcadia, Greece in 1882. He studied at the universities of Athens (Ph. D), Jena and Gottingen and was appointed to his present position in 1918. Dr. Sakellariou has also been a professor of theoretical mechanics at the Greek Naval Academy. A leader in mathematical education in Greece as well as a prominent research mathematician, Prof. Sakellariou is the president of the Greek Mathematical Society and an editor of the Bulletin of thet Society. He has written many textbooks ranging from Practical Arithmetic to Lectures on Theoretical Mechanics, as well as a number of papers on the calculus of variations and differential geometry.

A PROOF SCHEMA FOR A CLASS OF THEOREMS

Gene F. Rose

One may encounter in mathematics a class of theorems whose proofs have a common assential core. If one can discover this core, then one general proof can be given that will encompass all the theorems of the class. It is our purpose to examine such a case.

Consider the following two theorems, as typical of a class:

i) THEOREM. *Let f be any function of a real variable, and E any bounded closed point set in the linear continuum. If f is continuous on E, then f is bounded on E.*

ii) THEOREM. (HEINE-BOREL THEOREM.) *Let E be any bounded closed point set in the linear continuum. Map each point x of E into an open interval $U(x)$ which covers x. There is a finite subset of $\{U(x)\}$ which covers E.*

Proofs of these theorems can be given along the following lines: assume the result false, cover E by a closed interval E_0 and then apply the process of repeated bisection of E_0 to arrive at a contradiction. Such proofs have features which recur frequently in analysis. It seems natural to inquire to what extent the proofs are independent of characteristics peculiar to the individual theorems. Our subsequent analysis will reveal that, in each case, the proof stands on the fact that the theorem is true in a neighborhood of each point of E and that, if it is true on each of two subsets of E, then it is true on their sum. Furthermore, E may belong to a more general context than that of metrizable spaces. Thus, in Theorem 1, E is a subset of a topological S-space [1, p. 40][1]; i.e. a topological space R such that every neighborhood of an arbitrary point of R contains the closure of a neighborhood of the same point, and R contains a basis having not more than a countable number of open sets. This broadening of context entails the generalization from bounded closed sets E to compact sets [1, p. 42]. It is easy to establish that the n-dimensional Euclidean space R^n with its usual topology is an S-space (cf. [1, p. 32, Fxamples 13 and 14]); furthermore, as a result of the Bolzano-Weierstrass Theorem (cf. [2, Theorem 13.3, p. 38]), the bounded closed sets of R^n are compact. Hence the cited examples indeed fall within the context of Theorem 1.

THEOREM 1. *Hypothesis: R is an S-space; E is a compact subset of R; $P(X)$ is any propositional function whose variable X ranges over all subsets of R;*

(1) *for some neighborhood U of an arbitrary point of E, $P(EU)$;*

(2) *for all subsets U and V of E, $P(U)$ and $P(V)$ imply $P(U + V)$.*

Conclusion: $P(E)$:

PROOF. From (2), it is easy to prove by induction on r that, if

[1] Numbers in brackets refer to the bibliography at the end of the paper.

$P(X)$ is true on each of a finite number r of subsets of E, then it is true on their sum. Now, by (1), there exists a system of open sets, whose sum covers E, such that $P(X)$ is true on the intersection of E and each member of the system. From this system we can select [1, Theorem 7, p. 43] a finite subsystem $U_1, \cdots, U_r$ whose sum covers E. Inasmuch as $P(EU_j)$ for all j, we have $P(EU_1 + \cdots + EU_r)$. Because $EU_1 + \cdots + EU_r = E(U_1 + \cdots + U_r) = E$, we conclude that $P(E)$.

As another example of the application of Theorem 1, let us consider the following proposition $Q(E)$ regarding a function f continuous on the bounded closed point set E in the complex plane.

$Q(E)$: *f is uniformly continuous on E.*

It is expedient to treat a propositional function $P(X,\epsilon)$, where X ranges over sets in the complex plane and ϵ serves as a parameter ranging over rational numbers.

$P(X,\epsilon)$: *There is a positive δ such that, for all z_0 of X and all z_1 of E, $|z_1 - z_0| < \delta$ implies $|f(z_1) - f(z_0)| < \epsilon$.*

Then $Q(E)$ is equivalent to the proposition

For all positive ϵ, $P(E,\epsilon)$.

The complex plane is an S-space and E is a compact set. Furthermore, for an arbitrary positive value of ϵ, conditions (1) and (2) can be shown to hold for $P(X,\epsilon)$ as the $P(X)$ of Theorem 1. Thus, $Q(E)$ is established.

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University of Wisconsin

THE EXTENSION OF A RECTANGULAR MATRIX OF CONTINUOUS FUNCTIONS

H. F. Mathis

1. Introduction. In a paper by the author [1] the following theorem was presented.

Theorem I. *If the elements* $a_{ij}(t)$ $(i = 1, \cdots, p < q;\ j = 1, \cdots, q)$ *of a* $p \times q$ *rectangular matrix* A *are continuous functions of* n *variables* $t_1, \cdots, t_n$ *on a closed region* R *which is homeomorphic to a closed n-cell and* A *has maximum rank everywhere on* R*, then there exist polynomials* $b_{kj}(t)$ $(k = p + 1, \cdots, q)$ *in* t *such that the matrix*

$$M = \begin{pmatrix} a_{ij}(t) \\ b_{kj}(t) \end{pmatrix}$$

is non-singular everywhere on the closed region R.

The author has recently discovered that essentially the same theorem has been proved by Wazewski [2]. The same results can also be obtained from the theory of fibre spaces.

In this paper it will be shown that a similar theorem is also true in the case of a matrix whose elements are continuous functions of two variables (x,y) defined on a closed domain R bounded by a finite number of simple closed curves. This new theorem will be called Theorem II.

This theorem could also be proved using the theory of fibre spaces. The following proof does not require that the reader be familiar with more than a few of the most elementary topological concepts.

2. Proof of Theorem II. First consider the case where $q - p \geq 2$. If there exist functions $\beta_{mj}(x,y)$ $(m = p + 1;\ j = 1, \cdots, q)$ which are continuous solutions of the equations

$$(1) \qquad \sum_{j=1}^{q} a_{ij}(x,y)\beta_{mj}(x,y) = 0 \quad (m = p + 1;\ i = 1, \cdots, p)$$

and do not vanish simultaneously on R, the matrix

$$B = \begin{pmatrix} a_{ij}(x,y) \\ \beta_{mj}(x,y) \end{pmatrix}$$

has maximum rank everywhere on R. It will be shown how these functions can be found.

Let the outer-boundary of R be denoted by C_0 and the inner boundaries by $C_1, \cdots, C_\nu$. Let the order of the inner boundaries be chosen so that by making cuts between C_1 and C_2 on a simple curve c_1, between C_2 and

C_3 on a simple curve c_2, ..., and between C_ν and C_0 on a simple curve c_ν the region R is rendered simply connected. Let this new region be denoted by R'.

At any point (x_0, y_0) on R at least one of the $p \times p$ determinants of the matrix A does not vanish in a neighborhood of (x_0, y_0). Consequently $C_1, \ldots, C_\nu, c_1, \ldots, c_\nu$ can be divided into a finite number of closed segments s_δ on which at least one $p \times p$ determinant of the matrix A does not vanish. If the proper $q - p$ β's are arbitrarily chosen on one of the segments s_δ so that they are continuous, the remaining β's will be uniquely determined and continuous on s_δ. At any point (x_0, y_0) on R, more than one $p \times p$ determinant may be different from zero and corresponding to each of these determinants there is a different set of arbitrary β's. If any one set of these arbitrary β's vanish simultaneously at (x_0, y_0), all of the β's must vanish at this point.

Starting with any one of the segments s_δ on C_1, denote it by s_1 and choose the arbitrary β's for this closed segment so that they do not vanish simultaneously on s_1. Next choose one of the segments which has an end point in common with s_1 and denote it by s_2. Since the arbitrary β's for s_2 cannot vanish simultaneously at the common point between s_1 and s_2, they can be extended as constants on s_2. When the arbitrary β's for s_2 have been extended in this manner, the solutions for equations (1) are uniquely determined, continuous and do not vanish simultaneously on $s_1 + s_2$. This process can be continued for all of the segments s_δ on C_1 except the last one which will be denoted by s_r. The end-point values of the arbitrary β's for s_r have been fixed by this process so that all of them do not vanish at either of the end points. Since $q - p \geq 2$, there are at least two arbitrary β's for s_r. Therefore it is always possible to choose the arbitrary β's for s_r so that they are continuous and do not vanish simultaneously on s_r. Continuing this process it is possible to find the desired solutions of equations (1) everywhere on $C_1, \ldots, C_\nu, c_1, \ldots, c_\nu$.

There exists a homeomorphic mapping $(u,v) = f(x,y)$ of the interior of the region R' onto the interior of the unit square S with one corner at the origin and two of its sides along the u and v axes. This mapping can be chosen so that $C_1, \ldots, C_\nu, c_1, \ldots, c_\nu$ are mapped on the u and v axes and C_0 is mapped on the other two sides of the square. Each of the curves $c_1, \ldots, c_\nu$ will be mapped on two disjoint intervals on the boundary of the square.

The matrix $D = (a_{ij}(f^{-1}(u,v)) = \alpha_{ij}(u,v))$ has maximum rank everywhere on S. If there exist functions $\gamma_{mj}(u,v)$ which are continuous solutions of the equations

$$(2) \qquad \sum_{j=1}^{q} \alpha_{ij}(u,v)\gamma_{mj}(u,v) = 0 \quad (m = p+1;\ i = 1, \ldots, p)$$

and do not vanish simultaneously on S, the functions $\beta_{mj}(x,y) = \gamma_{mj}(f(x, y))$ are the desired solutions of equations (1).

Let $\gamma_{mj}(u,v) = \beta_{mj}(f^{-1}(u,v))$ for the points on the u and v axes, where the functions $\beta_{mj}(x,y)$ are the solutions of equations (1) along $C_1, \ldots, C_\nu, c_1, \ldots, c_\nu$ which were determined previously. The square S can be divided into a finite number of squares k_ϵ of equal size whose sides are parallel to the coordinate axes, so that for each square k_ϵ at least one $p \times p$ determinant of the matrix D does not vanish on the closed square k_ϵ. If these squares are properly ordered, the arbitrary γ's for each of them can be extended as constants along straight lines through the point $(u,v) = (-1,-1)$. Thus it is possible to find solutions of equations (2) which do not vanish simultaneously on S and consequently the desired solutions of equations (1).

Since the matrix B has maximum rank everywhere on R, the process can be continued until a $(q-1) \times q$ matrix E with rank $q-1$ is obtained. An additional row of continuous elements $\beta_{qj}(x,y)$ can be added to obtain a $q \times q$ matrix F. If the functions $\beta_{qj}(x,y)$ are chosen so that $\beta_{qj}(x,y)$ equals $(-1)^j$ times the determinant which is obtained from the matrix E by removing the jth column, the matrix F is non-singular everywhere on R. The functions $\beta_{kj}(x,y)$ $(k = p+1, \ldots, q;\ j = 1, \ldots, q)$ can be approximated by polynomials $b_{kj}(x,y)$ so that a matrix M is obtained which is non-singular everywhere on R.

3. General remarks. Since it is possible to relax some of the restrictions on the region R in Theorem I when $n = 2$, the following question arises. Is it always possible to weaken the restrictions on the region R if $n > 2$? Ważewski [2] gives the following simple example to show that the answer is no. It is impossible to extend the matrix $(t_1 t_2 t_3)$ on the surface $t_1^2 + t_2^2 + t_3^2 = 1$.

This example cannot be generalized to say that the matrix $(t_1 t_2 \ldots t_n)$ cannot be extended on the surface $t_1^2 + t_2^2 + \cdots + t_n^2 = 1$ for all values of $n > 2$. For example any matrix consisting of a single row of four elements which do not vanish simultaneously on a region R can be extended to become the matrix

$$\begin{bmatrix} a & b & c & d \\ b & -a & -d & c \\ c & d & -a & -b \\ d & -c & b & -a \end{bmatrix}$$

which is non-singular everywhere on R without any restrictions on the region R.

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The University of Oklahoma

COMPLEX VARIABLE THEORY

Edwin F. Beckenbach

1. *Introduction.* The beautifully coherent introductory theory of analytic functions of a complex variable has very, very much to offer the beginning graduate student.

Here, often for the first time, he gets a deep insight into the nature and workings of the *real* number system; this in part because a critical appraisal is inevitable in introducing the larger system of complex numbers; and also in part because, again often for the first time, the properties of the real number system are used here in a systematic way in constructing rigorous proofs of fundamental results in analysis. The rigor, surprisingly, is not particularly formidable; rather, the maturing student gets a sense of mathematical security in truly understanding his basic tools, and perhaps he wonders why some of the simple derivations were deferred beyond his sophomore work.

The theory of analytic functions of a complex variable is extensively interrelated with various other branches of mathematics. There is a welcome review of much of the student's former work in trigonometry, in analytic geometry, and in differential, integral, and advanced calculus, all in new and attractive settings. The complex number system is one of the most important systems in modern algebra. In complex variable theory, the Fundamental Theorem of Algebra is a trivially simple result. The theory also is fundamental in the analytical theory of numbers, and enters strongly into such fields as the theory of series and the theory of differential equations.

The elements of point-set topology, which are here introduced and developed if this has not been done in a previous course, are extensively used. Also a few necessary results in combinatorial topology, such as the Jordan Curve Theorem (in accordance with which a simple closed plane curve divides the plane into exactly two parts), are introduced, giving the student a natural and inviting introduction to this attractive branch of mathematics.

There are applications to the mathematical theory of electricity, to the theory of fluid motion and aerodynamics, to potential theory, and to other branches of physics; also to cartography, in the theory of conformal maps.

It should not be surprising, then, that a course in the theory of analytic functions of a complex variable is almost universally prescribed for graduate students of mathematics, and that most graduate students of physics, meteorology, and engineering are strongly urged to take the course. The students, for their part, because of the large and coherent volume of valuable mathematics which they have enjoyed assimilating, often look back on the course as a high point of their graduate training.

2. *The complex number system.* The "complex" number system is just as real as the "real" number system, in that it can be realized graphically (as we shall see in the next section), and can be used in describing and deducing results concerning the physical world. A mathematician who proves a complex variable theorem might thereby establish a result in real variable theory, or perhaps might see his result applied in the design of an airfoil.

Complex numbers are usually written in the form $a + bi$, where a and b are real numbers. The equation

$$(1) \qquad a + bi = c + di \quad \text{means} \quad a = c \quad \text{and} \quad b = d;$$

thus *a single complex equation is equivalent to two real equations.* Further, addition and multiplication are performed in accordance with the rules:

$$(2) \qquad (a + bi) + (c + di) = (a + c) + (b + d)i,$$

$$(3) \qquad (a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

These are the only basic rules for operations with complex numbers.

The above is equivalent to saying that complex numbers are ordered pairs (a first, then a second) of real numbers (a,b), for which $(a,b) = (c,d)$ if and only if $a = c$ and $b = d$; further, addition and multiplication are defined by

$$(a,b) + (c,d) = (a + c,\ b + d),$$

$$(a,b)(c,d) = (ac - bd,\ ad + bc).$$

Thus, though we shall give it a physical interpretation in the next section, the symbol "i" may be considered as serving the sole purpose of distinguishing between the first and second components of the complex number; nevertheless, the notation $a + bi$ is highly preferable, for with it we have only to remember that operations with complex numbers are just "ordinary" operations, with the single curious additional rule that $i^2 = -1$.

The complex numbers are not ordered: we do not say, for instance, that $2 + 3i$ is less than $4 + 5i$. Otherwise, in accordance with the above rules for equality, addition, and multiplication, *the complex numbers satisfy the same algebraic laws as the real numbers.* Thus from the law of commutation for addition of real numbers we have

$$a + c = c + a \quad \text{and} \quad b + d = d + b,$$

whence by (1) and (2) we have the same law for the addition of complex numbers:

$$(a + bi) + (c + di) = (c + di) + (a + bi).$$

The complex number $a + 0i$ is denoted by a, for it is identified with the real number a; this of course is consistent with the rules (1) – (3). If $b \neq 0$, then (unfortunately) $a + bi$ is said to be *imaginary*; the number $0 + bi$, $b \neq 0$, is denoted by bi, and is called a *pure imaginary*. We say that a is the *real part*, and b the *imaginary part*, of $a + bi$, and write

$$a = \Re(a + bi), \qquad b = \Im(a + bi).$$

Can complex numbers be subtracted? This is the same as asking if the equation

$$(4) \qquad (c + di) + (x + yi) = (a + bi)$$

can be solved for $x + yi$. By the above rules (1) and (2), the complex equation (4) reduces to the two real equations

$$c + x = a, \quad d + y = b;$$

and by our knowledge of real numbers these equations have the *unique solution*

$$x = a - c, \quad y = b - d.$$

We write

$$(a + bi) - (c + di) = (x + yi) = (a - c) + (b - d)i.$$

The complex number $(-c - di)$ is called the *negative* of $c + di$.

In an analogous way, we get

$$(5) \qquad \frac{a + bi}{c + di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}\, i,$$

provided $c^2 + d^2 \neq 0$, so that *in the complex number system division except by "zero" is always possible and unique.* As an exercise you might answer the following question: What complex number is the reciprocal of $c + di$? We shall ask similar questions, without giving answers, throughout the rest of our discussion of complex variable theory.

Granted that the quotient exists, we can obtain (5) as follows:

$$(6) \qquad \frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}.$$

It should be noted that the above rules for subtraction and division are not *prescribed*; rather, the operations are *defined* in terms of addition and multiplication, and the rules are then *deduced* from the basic rules (1) – (3).

The *conjugate* of $\gamma = c + di$ is $\overline{\gamma} = c - di$. In (6) we made use of the fact that the product of a complex number and its conjugate is real. What can you say about the sum and about the difference of a complex number and its conjugate? You might show also that the conjugate of the sum of any two complex numbers is the sum of their conjugates, and might investigate similar rules for their difference, product, and quotient.

3. *Graphical representation of complex numbers.* We have observed that a complex number $\alpha = a + bi$ can equally well be represented by an ordered pair (a, b) of real numbers. But we are also familiar with the representation of points of the Euclidean plane by Cartesian coordinates (a, b). It is only natural, then, to represent the complex number $\alpha = a + bi$ by the point P having coordinates (a, b), as in Figure 1. Equally well, we might think of the complex number α as being represented by the *vector* from the origin to P.

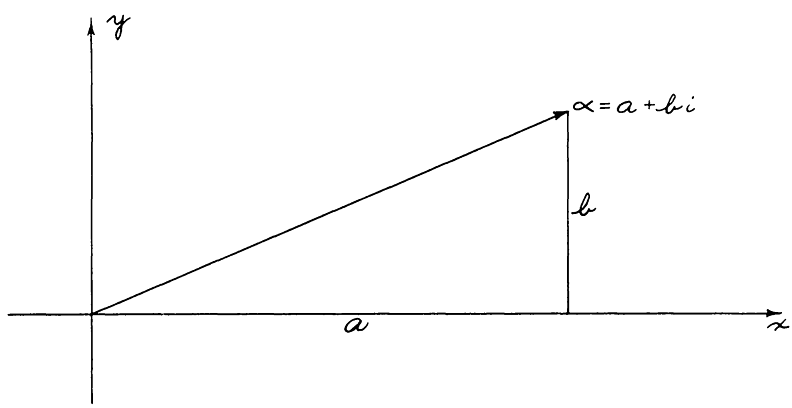


Figure 1

A plane on which complex numbers are thus represented is sometimes called a *Gauss plane* after the great German mathematician; again, this graphical representation of complex numbers also is often called an *Argand diagram* after a French mathematician who systematically used it. On this plane, for obvious reasons, the x- and y-axes are called the *real* and *imaginary axes*, respectively.

How do you locate the point representing the conjugate of a given complex number?

If the point P has polar coordinates (r, θ), $r \geq 0$, then in accordance with the definitions of the trigonometric functions we have

$$a = r \cos \theta, \qquad b = r \sin \theta,$$

whence

$$\alpha = a + bi = r(\cos \theta + i \sin \theta).$$

The value

$$r = (a^2 + b^2)^{1/2} = |a + bi| = |\alpha|$$

is called the *magnitude*, or *modulus*, or *absolute value*, of α, and θ is called its *argument*, or *amplitude*, or *angle*. It is to be noted that if $r = 0$ then θ is not determined by P, and that if $r \neq 0$ then θ is determined only to within an additive integral multiple of 2π radians; that is, we have

$$\alpha = a + bi = r[\cos(\theta + 2k\pi) + i \sin(\theta + 2k\pi)], \quad k = 0, \pm 1, \cdots.$$

It is easy to see that *complex numbers are added according to the vector law of addition*, as illustrated in Figure 2.

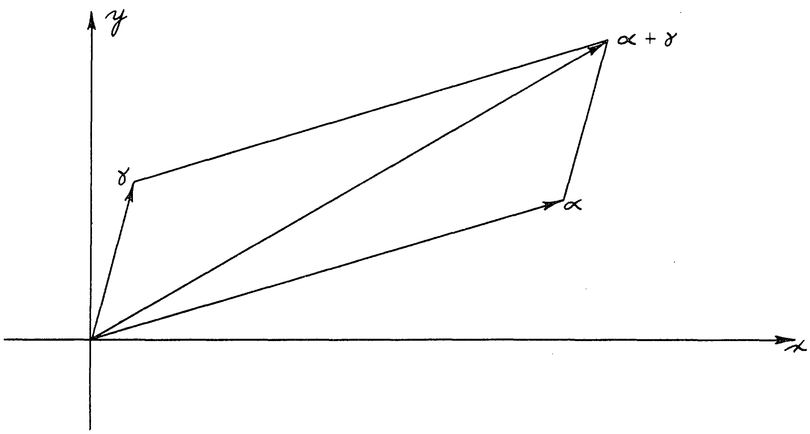


Figure 2

From our representation of ai, along with the relation $i^2 = -1$, we obtain the simple graphical interpretation that *multiplying by i means rotating the point, or vector, representing a complex number, about the origin through a positive angle of $\pi/2$ radians*. There is nothing imaginary about this!

Let us investigate, more generally, the geometric representation of the product of two complex numbers

$$\alpha = a + bi = r_1(\cos \theta_1 + i \sin \theta_1)$$

and

$$\gamma = c + di = r_2(\cos \theta_2 + i \sin \theta_2).$$

Multiplication of $a + bi$ by c gives a vector of length cr_1 in the direction θ_1, and by di a vector of length dr_1 in the direction $\theta_1 + \pi/2$. It follows almost immediately, as suggested by Figure 3, that the sum of these two component vectors of the product $(a + bi)(c + di)$ is a vector of magnitude $r_1 r_2$ in the direction $\theta_1 + \theta_2$. What familiar facts concerning triangles are involved in establishing this result?

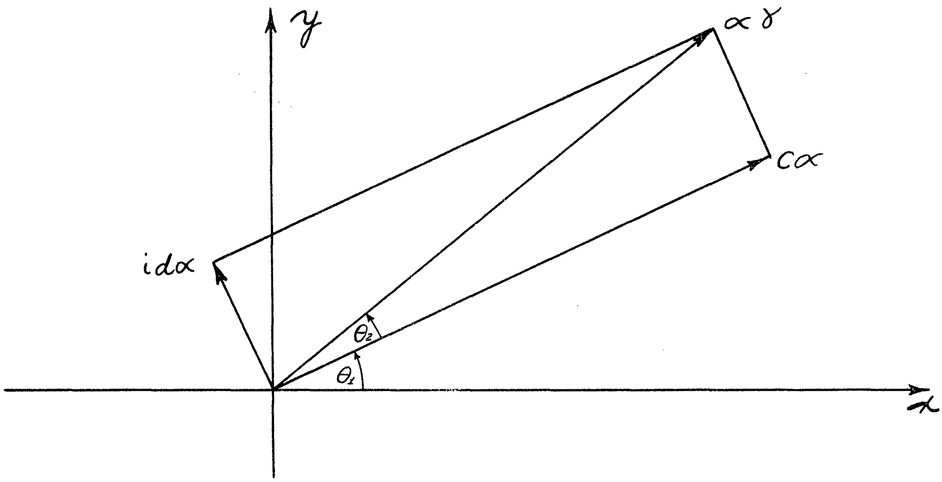


Figure. 3

In words, we have the rule: *the modulus of the product of two complex numbers is the product of their moduli, and the amplitude of the product is the sum of their amplitudes.*

The above result can be checked by means of the trigonometric identities

$$\cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2,$$

$$\sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2.$$

Thus

$$[r_1(\cos\theta_1 + i\sin\theta_1)][r_2(\cos\theta_2 + i\sin\theta_2)]$$

$$= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)$$

$$+ i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$$

$$= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)].$$

By mathematical induction, then, we have

$$[r_1(\cos\theta_1 + i\sin\theta_1)]\cdots[r_n(\cos\theta_n + i\sin\theta_n)]$$

$$= r_1\cdots r_n[\cos(\theta_1 + \cdots + \theta_n) + i\sin(\theta_1 + \cdots + \theta_n)];$$

that is,

(7) $$\prod_{j=1}^{n} r_j(\cos\theta_j + i\sin\theta_j) =$$

$$\left[\prod_{j=1}^{n} r_j\right]\left[\cos \sum_{k=1}^{n} \theta_k + i \sin \sum_{k=1}^{n} \theta_k\right],$$

which we call the *generalized De Moivre's formula.*

As a corollary of the rule for multiplying complex numbers, we have the result that the reciprocal of $r(\cos\theta + i\sin\theta)$ is $(1/r)[\cos(-\theta) + i\sin(-\theta)]$.

A graphical construction for the product of complex numbers is shown in Figure 4. You should study through this and justify it.

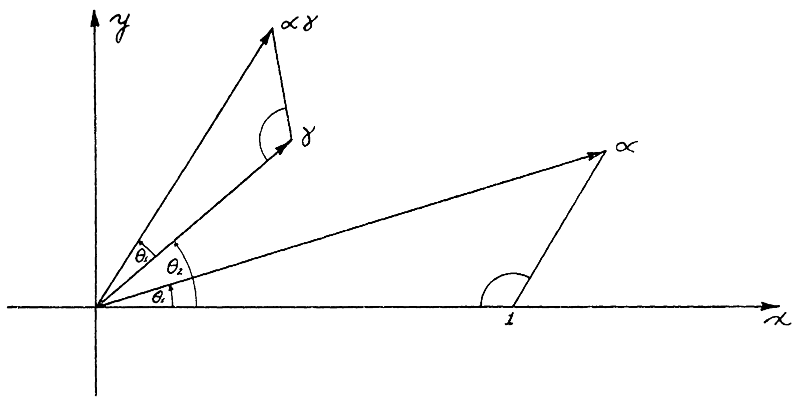


Figure 4

We could similarly give constructions for differences and quotients of complex numbers. But it is sufficient that we give the construction of the negative and the reciprocal of a complex number. Why? These constructions are shown in Figures 5 and 6, respectively.

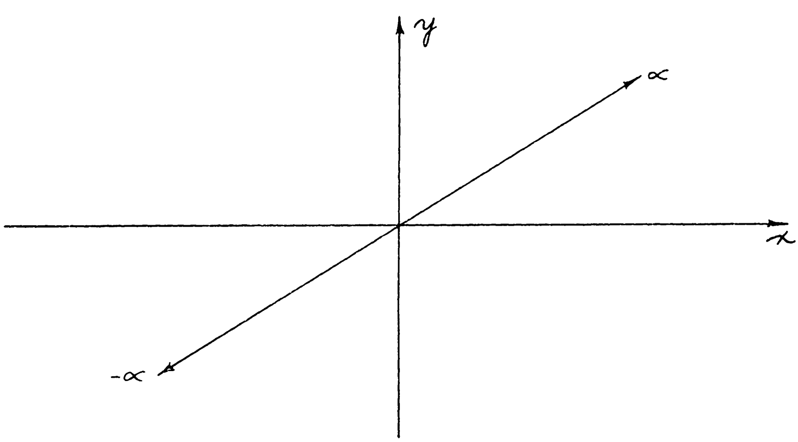


Figure 5

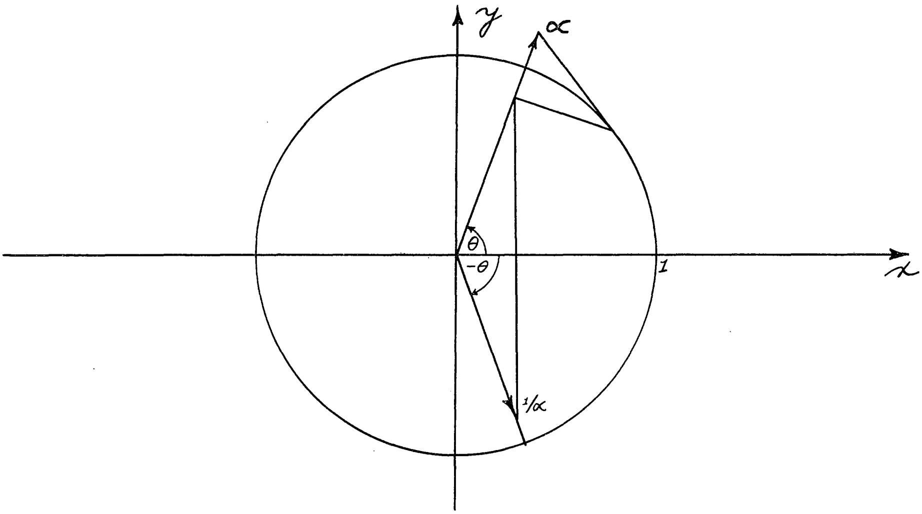


Figure 6

From the triangle inequality and Figure 2, it follows that

$$|\alpha| - |\gamma| \leq |\alpha \pm \gamma| \leq |\alpha| + |\gamma|.$$

Under what conditions do the signs of equality hold? These results actually could be obtained analytically as well as geometrically.

From the rule for the multiplication of complex numbers, we have

$$|\alpha\gamma| = |\alpha| \cdot |\gamma|;$$

also, if $\gamma \neq 0$, we have

$$|\alpha/\gamma| = |\alpha| / |\gamma|.$$

We easily obtain

$$|\alpha - \gamma| = [(a - c)^2 + (b - d)^2]^{1/2},$$

whence we have the important rule that *the absolute value of the difference of two complex numbers is equal to the distance between the points representing the numbers.*

If on a circle with center at the origin and radius $r > 0$ we plot n points equally spaced about the circle, starting with an angle θ as illustrated in Figure 7, we have points representing the complex numbers

$$\alpha_k = r[\cos(\theta + 2k\pi/n) + i \sin(\theta + 2k\pi/n)], \quad k = 0, 1, \cdots, n - 1.$$

Now, by (7), the nth power of α_k is given by

$$(\alpha_k)^n = r^n[\cos(n\theta + 2k\pi) + i \sin(n\theta + 2k\pi)] = r^n(\cos n\theta + i \sin n\theta),$$

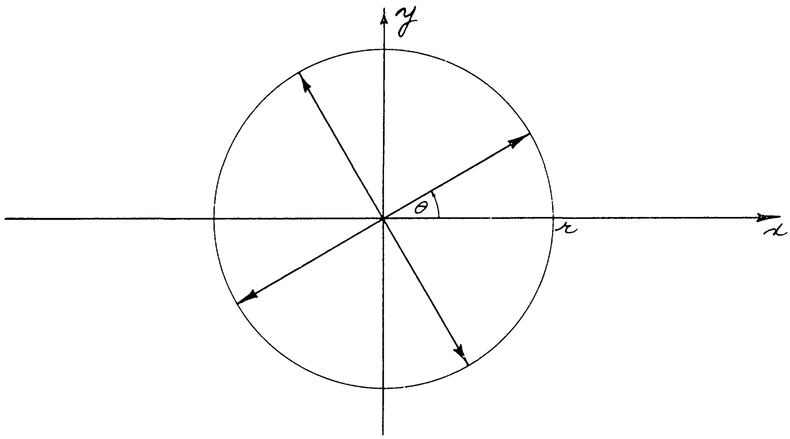


Figure 7

which is the same for all k. This process, applied in reverse, shows that *any complex number other than zero has exactly n distinct* nth *roots,* and shows how they are situated around a circle. Can you find the three cube roots of 1, and the four fourth roots of -1?

Again, if we expand the left-hand member of *De Moivre's formula,*

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta,$$

by the binomial formula, and apply (1), we obtain expressions for $\cos n\theta$ and $\sin n\theta$ in terms of $\cos \theta$ and $\sin \theta$. For example, we get

$$\cos 3\theta = \cos^3\theta - 3 \cos \theta \sin^2\theta, \quad \sin 3\theta = 3 \cos^2\theta \sin \theta - \sin^3\theta.$$

While these real expressions could be obtained without use of the complex number system, they could hardly be obtained more easily or elegantly.

4. *Stereographic project; the extended plane.* If you have studied Projective Geometry, you know that to the finite (though unbounded) Euclidean plane we adjoin, in each direction, an ideal "point at infinity" to obtain the projective plane; the totality of points at infinity we call the "line at infinity" in the plane. Why do we do this? There is no compulsion in the matter, but only a desire to avoid exceptional statements and to reduce our mathematical discipline to what appears to be a more unified form.

As we shall see, for the transformations of *Complex Variable Theory* a different convention about "infinity" serves a similar purpose.

In three-dimensional Euclidean space, consider the *unit sphere* S,

$$\xi^2 + \eta^2 + \zeta^2 = 1, \tag{8}$$

and let N denote its "North Pole," (0, 0, 1). The line joining N with any other point P of S meets the (ξ, η)-plane Π in a point P'. For simplicity of notation, in referring to P' we shall label the ξ- and

η-axes by x and y, respectively, and shall omit the vanishing third coordinate; thus P has coordinates (ξ, η, ζ), while P' has coordinates (x, y); see Figure 8.

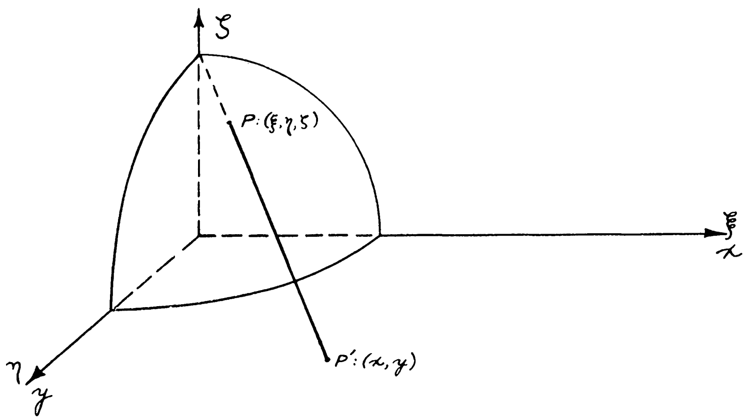


Figure 8

The correspondence thus set up between the points P of S and the points P' of Π is called a *stereographic projection* of S on Π. Under this transformation, what points of Π correspond to the points of the "Southern Hemisphere" of S? of the "Northern Hemisphere"? Are there any points which correspond to themselves?

But note that *there is no point of the Euclidean plane which corresponds to the point N on S.* We complete the one-to-one correspondence between the points of S and the points of Π by *defining* a single ideal *point at infinity* on the *complex plane* Π to correspond to N.

Most familiar plane maps of the Earth have the very useful property of being *conformal.* The above stereographic projection of S on Π has this same property: if two curves on S intersect at a certain angle θ (that is, if the angle between their tangents at the point of intersection is θ), then their image curves on Π necessarily intersect *at the same angle θ.* Now it happens that the principal transformations of Complex Variable Theory, which we designate by equations like $w = f(z)$ where z and w are complex variables, are *conformal plane transformations.* Under stereographic projections, to the plane variables z and w there correspond spherical variables Z and W, and the conformal transformation $w = f(z)$ *induces* a conformal relation between Z and W.

In some ways it is preferable to think thus of Complex Variable Theory in terms of conformal spherical maps rather than in terms of conformal plane maps. The unit sphere (8), when used in this way as range for a point corresponding to a complex variable, is called a *Riemann sphere.*

One item of caution: while in Complex Variable Theory we speak of "the point at infinity," and even write $f(\alpha) = \infty$, $f(\infty) = \beta$, or $f(\infty) = \infty$, we always imply the use of the symbol ∞ *only relative to the limiting processes of analysis* (which can best be visualized relative to the Riemann sphere). The rules (1) – (3) *do not apply* to this symbol; in Complex Variable Theory we do not even have $\infty + \infty \equiv \infty$.

5. *Loci; sets of points; topology.* For $z = x + iy$, the loci $|z| = 1$ and $|z - i| \leq 2$ are respectively a circle and a circle plus its interior. What are the centers and radii of these circles?

Since $\Re(z^2) = x^2 - y^2$, the locus $\Re(z^2) = 1$ is a hyperbola.

You could make an algebraic simplification to show that the locus of

$$\left|\frac{z - 1}{z + 1}\right| = 1$$

is the imaginary axis; but you can also obtain this result from geometric considerations.

For a finite z_0, the locus $|z - z_0| < \epsilon$, $\epsilon > 0$, is called an ϵ-*neighborhood* of z_0; the locus $|z| > M$, $M > 0$, is called an *M-neighborhood* of ∞. Corresponding *deleted neighborhoods* are the loci $0 < |z - z_0| < \epsilon$ and $M < |z| < +\infty$. (Note that this $+\infty$ is not the ∞ of Complex Variable Theory.)

A point z_0 is an *accumulation point* of a set E of points provided that each deleted neighborhood of z_0 contains at least one point of E. It follows that each neighborhood of an accumulation point of E contains an infinitude of points of E. Why?

A point z_0 is an *interior point* of E provided there is a neighborhood N of z_0 such that each point of N is a point of E.

A set E of points is *open* provided each point of E is an interior point of E, and a set E is *closed* provided each accumulation point of E is a point of E. Thus one of the half-planes $\Im(z) > 0$ and $\Im(z) \geq 0$ is open but not closed, while the other is closed but not open. The entire complex plane is both open and closed, while the set of points $1/n$, $n = 1, 2, \ldots$, is neither open nor closed.

A set E is *compact* provided every infinite subset of points of E has an accumulation point which is a point of E. Thus the set $|z| < 1$ is not compact, but the set $|z| \leq 1$ is compact. While the Euclidean plane is not compact, *the extended complex plane is compact.*

A continuous complex function of a real variable, of the form

$$(9) \qquad z = x + iy = z(t) = x(t) + iy(t), \qquad a \leq t \leq b,$$

is said to determine a *path* C. If the path is of finite length, it is said to be *rectifiable*. Clearly, a rectifiable path might intersect itself or retrace part of itself. If $z(a) = z(b)$, we say that the path C is *closed*.

A one-to-one and continuous (topological) image of the unit interval $0 \leq t \leq 1$ is called a *simple arc* or a *Jordan* arc, while a topological image of the unit circle $|z| = 1$ is called a *simple closed curve* or a *Jordan curve*. Each is the locus of a path.

An open set E is *connected* provided E admits no partition into two non-intersecting non-null open sets. A non-null connected open set is called a *domain*; the range of the independent variable for a function of a complex variable is usually taken to be a domain.

A domain D is *simply connected* provided every Jordan curve in D admits a continuous deformation in D to a point of D. Thus $|z| < 1$ is simply connected; the domain $1 < |z| < 2$ is *doubly connected*. You should consider the Riemann sphere in deciding whether or not the domains $|z| > 1$ and $1 < |z| < +\infty$ are simply connected.

The above notions and results, along with the Jordan Curve Theorem which we mentioned in the Introduction, include some of the more important aspects of point-set topology and of combinatorial topology with which we are concerned in Complex Variable Theory.

6. *Complex functions; Riemann surfaces*. Quite generally, if to each element z of a set D of elements of any sort whatsoever there correspond one or more elements w of a set E, we say that w is a *function* of z, and denote the relation by such an equation as

$$(10) \qquad w = f(z), \qquad z \text{ in } D.$$

A current practice is to speak of the "points" z and w of the "spaces" D and E, regardless of their actual nature: the point w_0 might be the person occupying the chair, or point, z_0 in the space of chairs at a certain dinner party.

We shall be concerned with the case in which D and E are sets of complex numbers, so that $w = f(z)$ is a complex function of a complex variable. Examples of such functions are $w = z^2$ and $w = |z|$ (though the latter is also, more precisely, a real function of the complex variable z).

For a complex function of a complex variable, equation (10) can be written as

$$(11) \qquad w = u + iv = f(z) = f(x + iy) = u(x,y) + iv(x,y),$$

so that *a complex function of a complex variable may be regarded as an ordered pair (u,v) of real functions of two real variables (x,y).*

Since there are four real variables $(x,y;\ u,v)$ involved in (11), in order to obtain a graphical representation of such a function, analogous to the familiar representation of a curve $y = f(x)$ in the Cartesian plane, we would need a four-dimensional real space. We actually achieve a representation either by using one sheet of graph paper for the z-plane and another for the w-plane, or by using just one sheet with one axis serving for both x and u and the other for both y and v. Riemann spheres might be substituted, especially if the graphs include the point at infinity. In any case, corresponding points z and w are labeled similarly in order to picture the functional relation, which might be regarded either as a physical mapping or as a mathematical transformation.

The following example illustrates the above discussion and also introduces an additional consideration. For the function

$$w = z^2, \tag{12}$$

we write

$$z = r(\cos\theta + i\sin\theta), \qquad w = \rho(\cos\phi + i\sin\phi);$$

then by De Moivre's formula we have

$$\rho(\cos\phi + i\sin\phi) = r^2(\cos 2\theta + i\sin 2\theta),$$

so that $\rho = r^2$ and $\phi = 2\theta + 2k\pi$, where k is an integer. Thus if the variable point z describes the circle $|z| = r$, $0 < r < +\infty$, at a constant rate σ, then the corresponding point w continuously describes the circle $|w| = r^2$, but (since $d\phi/d\theta = 2$) at twice the rate σ! When the point z has gone halfway around its circle, the point w has gone completely around. And when z has gone all the way around, w has correspondingly gone around exactly twice; both points have simultaneously returned to their original positions. Letting r vary from 0 to $+\infty$, we see just how to every value w there correspond exactly two values z_1 and z_2, *except* that to $w = 0$ and $w = \infty$ there correspond the unique values $w = 0$ and $w = \infty$, respectively. We say that the function $w = z^2$ takes on the values 0 and ∞ with *multiplicity two* at $z = 0$ and ∞, respectively.

We might describe the above situation in terms of Riemann spheres as follows: To the points of a certain meridian semicircle on the z-sphere there correspond the points of a meridian semicircle on the w-sphere. As the former meridian revolves and covers its sphere exactly once, the latter revolves twice as fast and covers its sphere twice. Since both meridians are now in their original positions, we consider each of the surfaces which they have generated as being sewn together along its initial and terminal edges – though the manner in which the two-sheeted *Riemann surface* over the w-sphere has managed to pierce itself must remain, physically, something of a mystery!

The above device has served the very desirable purpose of *uniformizing* the relation between z and w. While it is not true that there is a one-to-one relation between *z-values* and *w-values* for the function (12), there is nevertheless a one-to-one relation between the *z-points* on the single-sheeted Riemann surface over the z-sphere on the one hand, and the *w-points* on the two-sheeted Riemann surface (with *branch-points* at $w = 0$ and $w = \infty$) on the other. (The Riemann surfaces might equally well have been considered over the z- and w-planes.)

If we designate a particular meridian on the above two-sheeted

Riemann surface as *branch-cut*, and consider that it separates the surface into two distinct sheets, such that when a variable point crosses this meridian it passes from one sheet to the other, then to each of these sheets corresponds half of the z-plane (if z_0 is in one half, then $-z_0$ is in the other), and we are led to a consideration of two single-valued *branches* of the double-valued function $z = w^{1/2}$. The branch cut might actually be any simple curve from $w = 0$ to $w = \infty$.

You might enjoy investigating Riemann surfaces for the functional relation defined by

$$w^2 = z^3.$$

It follows from the Fundamental Theorem of Algebra that the function

$$w = a_0 z^n + a_1 z^{n-1} + \cdots + a_n, \qquad n \geq 1,$$

where the a_j are real or complex constants with $a_0 \neq 0$, takes on each finite value w exactly n times (if multiple roots are properly counted) as z ranges over the finite plane; w has an n-fold infinity (pole) at $z = \infty$. More generally, a *rational function*

$$w = \frac{a_0 z^n + \cdots + a_n}{b_0 z^m + \cdots + b_m}, \quad a_0 \neq 0, \ b_0 \neq 0,$$

with real or complex coefficients and of degree $N = \max(n, m) \geq 1$, takes on each value w in the *closed* complex plane exactly N times as z ranges over the closed plane; *this result well illustrates the unifying advantage of our convention about the point at infinity in the complex plane.*

7. *Continuity; differentiation.* In considering the graph of the function (12) we tacitly used the fact that this function is continuous.

We say that the function $w = f(z)$, defined in a domain D, is *continuous* at a point z_0 of D provided that to any neighborhood N of $w_0 = f(z_0)$ there corresponds a neighborhood N' of z_0 such that for each z in N' the point $w = f(z)$ is in N. Graphically, what does this mean? For finite z_0 and w_0, can you express the condition in ϵ, δ notation? What alteration must you make in the description if z_0 or w_0 or both are infinite? Can you express the condition in terms of a limit?

If $f(z)$ is continuous at each point of D, we say that $f(z)$ is *continuous in D*.

In writing

$$\lim_{z \to z_0} f(z) = w_0,$$

we should think in terms of neighborhoods, as above, rather than using such expressions as, "independent of the mode of approach," or "independent of the path of approach." In fact, "independent of the (straight) line of approach" is not a sufficient condition for continuity, as the following example shows.

Let $w = f(z) = f(x + iy)$ vanish identically on the left-hand half-plane $\Re(z) \leq 0$, and also on the entire real axis $\Im(z) = 0$. Let $f(z)$ take the value 1 at each point, except the origin, on the parabola $x = y^2$; and let $f(z)$ vary linearly from 0 to 1 on each horizontal line segment from the imaginary axis $\Re(z) = 0$ to the parabola, and also on each line segment from the origin to the parabola. It is easy to see intuitively that $f(z)$ is continuous at each point of the finite plane other than the origin, that

$$\lim_{z \to 0} f(z) = 0$$

for every linear approach, but that

$$\lim_{\substack{z \to 0 \\ x = y^2}} f(z) = 1.$$

You can readily check analytically that the function

$$g(z) \equiv g(x + iy) \equiv \frac{2xy^2}{x^2 + y^4}, \quad z \neq 0, \tag{13}$$

$$g(0) = 0,$$

has the same anomalous behavior in the neighborhood of the origin as has the preceding function $f(z)$. Namely, for linear approaches ($y = mx$ or $x = 0$) we have

$$\lim_{z \to 0} g(z) = 0; \tag{14}$$

but

$$\lim_{\substack{z \to 0 \\ x = y^2}} g(z) = 1. \tag{15}$$

Just as the above definition of continuity of a complex function of a complex variable is formally the same as the familiar definition of continuity of a real function of a real variable, the definition of derivative also is the same in the two cases. Thus for a point z_0 of the domain of definition of a complex function $f(z)$, we investigate

$$\lim_{\substack{z \to z_0 \\ (z \neq z_0)}} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z}, \tag{16}$$

and if this limit exists, we say that $f(z)$ has a *derivative* at z_0, which we denote by $f'(z_0)$ or $df/dz|_{z=z_0}$.

As in the definition of continuity, plane neighborhoods are involved also in the limiting process in (16). Thus for the function

$$h(z) \equiv zg(z),$$

where $g(z)$ is given by (13), if we take $z_0 = 0$ we have

$$\frac{\Delta h}{\Delta z} = \frac{h(z) - 0}{z - 0} = \frac{zg(z)}{z} = g(z),$$

whence, as we have seen in (14) and (15), for all linear approaches we have

$$\lim_{\Delta z \to 0} \frac{\Delta h}{\Delta z} = 0,$$

while

$$\lim_{\substack{\Delta z \to 0 \\ x = y^2}} \frac{\Delta h}{\Delta z} = 1,$$

so that the function $h(z)$ does not have a derivative at the origin.

On the other hand, you can readily check that

$$\frac{dc}{dz} \equiv 0, \ \frac{dz}{dz} \equiv 1, \ \frac{dz^2}{dz} \equiv 2z, \ \frac{d(1/z)}{dz} \equiv -\frac{1}{z^2},$$

and so on. You can also check that if $f(z)$ has a derivative at z_0, then $f(z)$ must be continuous at z_0.

By taking separately $\Delta z = \Delta x$ and $\Delta z = i\Delta y$ (that is, by taking horizontal and vertical increments to z), and using (1), you can show that if a function $f(z)$, given by (11), has a derivative at a point, then the four partial derivatives $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, and $\partial v/\partial y$ exist at the point and satisfy the *Cauchy-Riemann partial differential equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{17}$$

at the point. Conversely, you can show by the law of the mean that if the partial derivatives are continuous and satisfy (17) throughout a domain, then the function (11) has a derivative at each point of the domain; actually, the assumption of continuity here is superfluous, but the proof then becomes quite difficult.

8. *Complex integration; Taylor series.* Let the function $f(z)$, given by (11), be continuous in a domain D, and let C be a rectifiable

path in D given by an equation of the form (9). We consider a sequence of divisions

$$a = t_0^{(n)} < t_1^{(n)} < \cdots < t_n^{(n)} = b, \qquad n = 1, 2, \ldots,$$

with

$$\lim_{n\to\infty} \max_{j=1}^{n} [t_j^{(n)} - t_{j-1}^{(n)}] = 0;$$

we choose values $\tau_j^{(n)}$ with

$$t_{j-1}^{(n)} \le \tau_j^{(n)} \le t_j^{(n)}, \qquad j = 1, 2, \ldots, n;\ n = 1, 2, \ldots;$$

and we form the sequence of sums

$$\sigma_n = \sum_{j=1}^{n} f[z(\tau_j^{(n)})][z(t_j^{(n)}) - z(t_{j-1}^{(n)})] = \sum_{j=1}^{n} f(\zeta_j)\Delta z_j.$$

It can be shown that

$$\lim_{n\to\infty} \sigma_n$$

exists and is independent of the choice of the $t_j^{(n)}$ and $\tau_j^{(n)}$ satisfying the above conditions; indeed, the limit is independent of the particular parametrization of C. We write

$$\lim_{n\to\infty} \sigma_n = \int_{z(a)}^{z(b)} f(z)\,dz. \tag{18}$$

You will note that (18) includes real integration as a special case. The integral (18) can be regarded as a complex Riemann-Stieltjes integral,

$$\int f[z(t)]\,dz(t),$$

and can also be broken up into real line integrals,

$$\int (u\,dx - v\,dy) + i\int (v\,dx + u\,dy).$$

One of the most important integrals in Complex Variable Theory is

$$\int_{|z|=r} \frac{dz}{z},$$

where the circle $|z| = r$ is traversed once in the positive (counterclockwise) direction; perhaps you can verify that the value of the integral is $2\pi i$.

You can show directly from the definition of integration that for any complex constant k we have

$$\int_{z'}^{z''} k\, dz = k(z'' - z'),$$

regardless of the rectifiable path of integration from z' *to* z''. Why is this equivalent to saying that we have

$$\oint_C k\, dz = 0$$

for any *closed* rectifiable path C?

The above result is a special case of the following fundamental theorem.

Cauchy Integral Theorem. *Let the function $f(z)$ have a derivative at each point of the finite simply connected domain D. Then we have*

$$\oint_C f(z)dz = 0$$

for each closed rectifiable path C in D.

Consequences of the Cauchy Integral Theorem are numerous and important.

Thus under the hypotheses of the Cauchy Integral Theorem we have the Cauchy Integral Formula:

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

provided C is a simple closed rectifiable path in D and z is in the interior of C. Remarkably, then, the values of $f(z)$ inside C are determined by its values on C!

When we apply Leibniz' rule for differentiating under the integral sign to the Cauchy Integral Formula, we obtain

$$f'(z) = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta,$$

and more generally

$$f^{(n)}(z) = \frac{n!}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta.$$

Thus the assumption that $f'(z)$ *exists* at each point of D has led to the conclusion that $f(z)$ has continuous derivatives of all orders throughout D.

Again, since the integral is independent of the path of integration, we may write

$$\int_{z_0}^{z} f(\zeta)d\zeta = F(z),$$

and it is easy to show that

$$\frac{dF(z)}{dz} = f(z). \tag{19}$$

Hence if $f(z)$ has a derivative at each point of the finite simply connected domain D, then $f(z)$ has a *primitive* $F(z)$ in D; that is, there is a function $F(z)$ satisfying (19) in D. The function $F(z)$ is unique to within an arbitrary additive constant.

Finally, if C is a circle with center z_0, and z is a point inside C, we use the geometric series

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0}\left[\frac{1}{1 - \left(\frac{z - z_0}{\zeta - z_0}\right)}\right]$$

$$= \frac{1}{\zeta - z_0}\left[1 + \frac{z - z_0}{\zeta - z_0} + \left(\frac{z - z_0}{\zeta - z_0}\right)^2 + \cdots\right]$$

and the Cauchy Integral Formula to obtain the *Taylor Series*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n. \tag{20}$$

Conversely, if a power series

$$\sum_{n=0}^{\infty} a_n (z - z_1)^n \tag{21}$$

converges for any value $z = z_2 \neq z_1$, then the series converges for every value in the circle $|z - z_1| < |z_2 - z_1|$, and the series converges uniformly in any smaller concentric circle. The series represents a differentiable function, whose derivative is obtained by differentiating the series term by term.

9. *Analyticity*. The theory of analytic functions of a complex variable was developed from different points of view by A. L. Cauchy and K. W. T. Weierstrass.

Cauchy considered the function $f(z)$ and its entire domain D of definition as being given. One assumes the existence of a derivative at each point of D and proceeds to develop properties of the function as discussed above.

Weierstrass considered a power series of the form (21) as being given. If the series converges for any value other than z_1, then it has a *circle of convergence* and is called a *function element*. Now the function can be re-expanded by (20) about any point z_0, and the new circle might (or might not) extend outside the original circle of convergence; see Figure 9. Continuing thus, we theoretically generate the function in its entire domain of analytic existence. (The extension is unique if possible.)

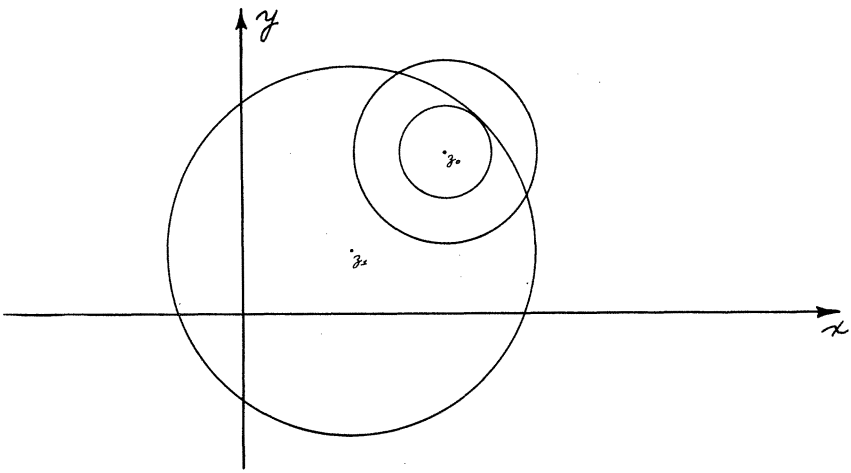


Figure 9

We have indicated in §8 that the two definitions are equivalent. In fact, we have indicated several possible equivalent definitions, or tests, of analyticity. We might briefly recapitulate.

Let $f(z)$, given by (11), be continuous in a finite simply connected domain D. We might test whether or not the function satisfies any one of the following properties.

A. (Analyticity test.) The function has a derivative at each point of E.

C. (Integral test.) The function satisfies

$$\oint f(z)dz = 0$$

for each closed rectifiable path C in D.

P. (Primitive test.) There is a function $F(z)$ such that in D we have

$$\frac{dF}{dz} = f(z).$$

CR. (Partial derivative test.) The partial derivatives $\partial u/\partial x$,

$\partial u/\partial y$, $\partial v/\partial x$, and $\partial v/\partial y$ (are continuous) and satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

throughout D.

S. (Series test.) The function can be represented by a convergent power series in any circle in D.

At various places in this paper we have indicated that we have the implications shown in Figure 10. Can you find, in particular, where we have shown that $P \to A$? A corollary of our implications is $C \to A$; this called *Morera's theorem.*

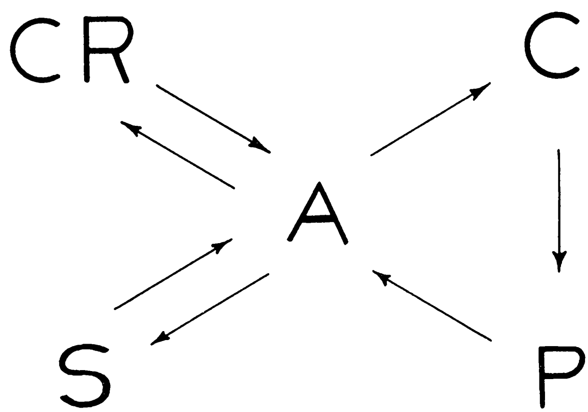


Figure 10

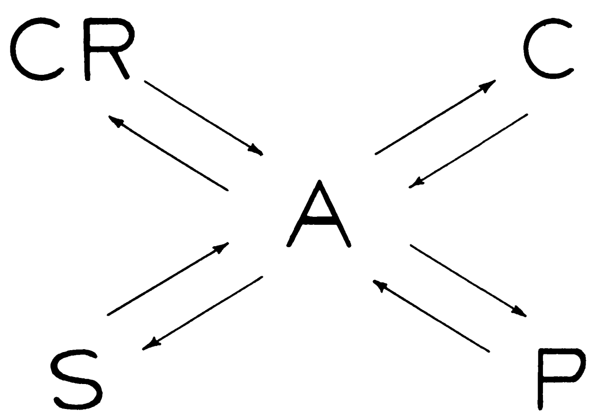


Figure 11

Why can the implications of Figure 10 be completed as in Figure 11? But note that if D is not a finite simply connected domain then some of our implications do not hold as

$$\int_{|z|=1} \frac{dz}{z} = 2\pi i$$

shows.

10. *Conclusion.* We have indicated only some of the basic notions and tools of Complex Variable Theory. The fascinating story goes on and on.

University of California, Los Angeles

UNIFIED FRESHMAN MATHEMATICS

S. E. Urner

The search for a "unified" approach to Freshman Mathematics appears to stem principally from two sources: (1) The philosophical, or theoretical, recognition that the cultural and disciplinary values of mathematics may be enhanced by such an approach; (2) The practical need for early introduction of the calculus in the training of prospective engineers and scientists. The pioneer effort in the text-book field seems to be the "Elementary Mathematical Analysis," by Young and Morgan – one of a series edited by Professor E. R. Hedrick. The copyright date is 1917. The authors say: "So much has been said in recent years in favor of a unified course in mathematics for freshmen that it seems desirable actually to try it out in practice. For this purpose a text-book is necessary. We do not believe that this text will solve the problem; the most we can hope for is that we have secured a first approximation." On the philosophical side, I quote: "The concept of *functionality* and the mathematical processes developed for the study of functions are precisely the things in mathematics that have most effectively contributed to human progress in more modern times; and the thinking stimulated by this concept and these processes is fundamentally similar to the thought which we are continually applying to our daily problems. By making the concept of a function fundamental throughout, we believe we have gained a measure of unity impossible when the year is split up among several different subjects." They also invite "advice and counsel as to the possible desirability of increasing the amount of calculus included in the first year," which on theoretical grounds they believed desirable. On this point, they conclude, "if the results are satisfactory, we could then take the next step with confidence."

One year later there appears "Unified Mathematics," by Karpinski, Benedict and Calhoun. They say, "the fundamental idea of the development is to emphasize the fact that mathematics cannot be artificially divided into compartments with separate labels, as we have been in the habit of doing, and to show the essential unity and harmony and interplay between the two great fields into which mathematics may be properly divided; viz., analysis and geometry." As to the calculus, however, they say, "No attempt has been made to introduce the terminology of the calculus, as it is found that there is ample material in the more elementary fields which should be covered before the student embarks upon what may properly be called higher mathematics. However, the fundamental idea of the derivative is presented and utilized without the new terminology."

These two books remained as pioneering landmarks, and they seem to have made little impression upon the educational practices of the times. From the point of view of actual influence and widespread use, the outstanding work has been that of Professor F. L. Griffin: "Introduction to Mathematical Analysis," first published in 1923. We quote him: "Under the traditional plan of studying trigonometry, college algebra, analytic geometry, and calculus separately, a student can form no conception of the character and possibilities of modern mathematics, nor of the relations of its several branches as parts of a unified whole, until he has taken several successive courses. Nor can he, early enough, get the elementary working knowledge of mathematical analysis, *including integral calculus*, which is rapidly becoming indispensable for students of the natural and social sciences."

The discussion will become more concrete if we place it in a framework of actual experience. I taught for nineteen years, 1931-1950, in Los Angeles City College. In the mathematics department, our piéce de resistance was the two-year course which duplicated the mathematics of the standard four-year engineering program. Normally, these students came to us with 1½ to 2 years of algebra, 1 year of geometry, and ½ year of trigonometry. We fed them one semester each of college algebra, analytic geometry, differential calculus, integral calculus; so that ordinarily it took an extra semester to do the "third course in calculus," without which they would be deficient upon transfer to the engineering school. Furthermore, the physics course the students were taking used calculus even in the first semester, and we gave them none before the sophomore year. This latter situation was a powerful stimulus to some sort of change. But there were other tensions. Our college aglebra classes were a headache. Review material, no matter how badly needed, is still review, and as such lacks inspirational power. And it seemed impossible to conduct these classes without large emphasis on review; while even the newer material seemed so similar to the old that the challenge of adventure was lacking. At any rate, the students seemed to regard the college algebra as a chore which had to be gotten out of the way before their real objective could be approached.

I am sorry to report that in lesser degree the analytic geometry met a similar fate. There was always the feeling that we were getting ready for something important, which in the meantime was being withheld. By this time a whole year has gone by, and we are still in the preparatory stages. And those students who for any reason do not go beyond one or two semesters leave with only a dim, fragmentary view of "what it is all about."

Shortly after Griffin's revised text came out in 1936, we adopted it. Certain things were at once evident. For example, our motivation problem was pretty well solved. In contrast to the traditional dragging of feet in college algebra, we began to hear such comments as "a fine course," "very interesting," "good stuff," etc. Of course, our difficulties didn't all disappear, nor did all our students become successful mathematicians. But we certainly had a better morale. And when the instructor read his final examinations at the end of the first semester, he couldn't help thinking how much more valuable to the student was the outcome, as compared with a semester of college algebra, in case his mathematical experience ended at that point.

If the prospective student had had only one year of algebra in high school, he was required to take a semester of intermediate algebra before he could go ahead; likewise, he had to take trigonometry, if he had not already had it. This was in spite of the fact that the text did not assume trigonometry. The idea was that most of the pre-engineers would have had trigonometry in high school, and that therefore we ought to limit ourselves in the main to certain aspects of the material - especially the analytical aspects, and applications which would seem new and interesting in view of the impact of the calculus. (Otherwise, we should have an echo of the college algebra problem.) But experience showed that there was also value in having the trigonometric material available for reference as needed by the individual student.

When we try to analyze the results in motivation, we can refer them to Professor Young's "concept of functionality;" or perhaps in more homely terms, ascribe them to the invigorating influence of the calculus. You have no doubt heard students say, "I never really learned algebra until I studied calculus." I have always interpreted this as a lesson in motivation. Perhaps along this path we can find the answer to the doubts of those who feel that the students we are discussing need more algebraic drill and experience before we can entrust to them the responsibilities of the calculus. Let us admit that there are some phases of the calculus which require only very simple algebraic manipulations. Why, then, can't we begin with these materials, meanwhile organizing our work so that the techniques and skills required later can be developed by work with new material? A bit later, I will give some examples to show what I mean.

It is worth noting that the course widely known as "Third Course in Calculus" is traditionally of a unified type, in that it includes solid analytic geometry and infinite series. The arrangement is no doubt due to the need for the solid analytics as a background for partial differentiation and multiple integration; and that for infinite series in preparation for expansion of functions; whereas there has been no opportunity to study these topics at an earlier stage. Many teachers will recognize also the value of placing the background material as near as possible to its application, lest it be lost in the interval between them.

Further, even within the first year of calculus, there is a trend toward internal unity. The rigid barrier between *differential* and *integral* calculus tends to break down; so that it is no longer a point of honor to withhold all integrals while the semester of derivatives is being absorbed.

There are text-books which, in the interest of sales appeal to many highly individualistic instructors, make the claim that the chapters "may be taken up in any order." They follow a "unit" plan like that which is now popular in some secondary circles. This seems to be a sort of "reductio ad absurdum" of the traditional separation of mathematics into self-contained, non-interrelated courses. By contrast, let us imagine a course in which each concept and technique is kept alive and working, once it has been introduced. Such a course would possess a unity beyond that which is implied in the mere combination of erstwhile distinct subjects into a single course. At once we admit that such an ambition represents an *ideal* which we cannot hope to attain in ultimate perfection. My proposition is that it represents a *goal*, which if left in sight and mind, will tend to vitalize our work, and to increase the serviceability and all-round availability of our mathematics to the student.

Right here we begin to sense the difficulties inherent in the enterprise we assume ourselves to have started. In the first place we were, most of us, brought up in the compartmental pattern. Hence this pattern seems natural and logical to us. We think only reluctantly and awkwardly in the new pattern. We may take three or four books, bind them in one volume, and call the result a unification. Or we may slice the books into sections and interweave them. The problems we publish are likely to be compilations and adaptations of the heritage of the race of text-book authors. Hence they contribute little to the ideal unity just described. And if these things, by some miracle, were not true of ourselves, the writers, we are quite sure they will be true of the users. Hence, if we progress too far toward our ideal, we are sure we shall find no market. Not only will this be sad, financially, but also we shall feel frustrated over the minuteness of our contribution to the progress of mathematical teaching.

In this connection, let me quote from an editorial in the "Scientific American" of February, 1924: "In teaching, more than in any other profession, there is temptation to fall into a rut-bound conservatism. The inherently self-perpetuating character of this profession is responsible for the conservatism in question. - - - The teacher of this generation is the student of the last generation; and when, as teacher, he deals with his students, he finds himself in exactly the same situation that his former teacher occupied in dealing with him. He may meet this in a manner highly inspirational, but only occasionally may we expect to find him developing sufficient initiative to question the fundamentals, and to wonder whether the manner in which his subject was taught to him is, after all, the one-and-only, God-given way to teach it." (The title of the editorial, by the way, is "The Unity of Mathematics").

But beyond these pessimisms lies the challenge! After all, the most compelling reason for being a mathematician is the challenge. And I have long been persuaded that of coordinate importance with mathematical research is research on the problem of how best we may improve the presentation of our subject - so that it may be as meaningful as possible to as many as we can get to listen; and so that those who listen may catch to the utmost the thrill and fascination which are there, though often concealed beneath a forbidding exterior!

To come down to earth again, our plan will have the dangers of its virtues; that is, the richness of new material is indeed a motivation, but there is danger that new ideas brought on too fast will lead to non-assimilation, confusion, indigestion. This danger emphasizes our need to watch the ideal goal - to see that each new concept is applied repeatedly as we proceed. There is also the matter of academic credits - traditionally based on quid pro quo recognition of standard compartmentalized units. I invite discussion as to how large this difficulty looms as against the benefits we hope to achieve. Perhaps it is mainly a transition problem - one of our growing pains. Certainly it arises only in connection with transfers between colleges. And may I suggest that we are often fooling ourselves in assuming that each unit of credit corresponds to a definite segment of mathematical achievement.

In order to escape from mere generalities, let us consider a few concrete examples. Since the basic approach is a *functional* one, we strive, by means of extensive graphical experience, to create a fundamental awareness of function-relationships. First we use tables of data, or tables constructed by the student from verbal directions. Then we introduce algebraic formulations, taking care to provide material which draws attention to fundamental things like radicals, exponents, multiple-values, division by zero, etc. But in all cases these matters remain subordinate to the main enterprise of the study of functions. We concern ourselves with interpolation, rate of change, maxima and minima, treating them all graphically at first. Then we adapt an algebraic viewpoint, studying first the linear function - not because the straight line is basic in geometry, but because the rate of change of the linear function is constant. We follow up with such notions as slope and intercept, and develop various forms of the equation of a straight line, with appropriate exercise material.

The quadratic function follows. By reducing the general quadratic to a square, and bringing in the technique of translation, we reduce the problem to the study of the function ax^2, and thus obtain a preliminary treatment of the parabola. Exercises may include work on the falling body problem, with the equation of motion assumed.

A bit later, we study implicit functions. It can be argued that these should be much further postponed, since their complications may obscure the broad fundamental concepts we are striving for. But we have promised to provide concurrently some algebraic experience, to compensate for our having plunged into calculus without further algebraic hurdles. So we use a variety of equations, including such equations as $x^2 - 3xy + 2y^2 - x + 4y - 2 = 0$, and $16x^3 - 4x^2y - 36xy^2 + 9y^3 = 0$.

As soon as we have differentiated x^n (n rational) and are provided with certain fundamentals, including the function-of-a-function formula, we are ready for a wide range of applications, and even for a fairly extensive integration section. In all this, there has been no need for a more extensive analytic geometry, dealing with the circle and the conics. There will still be time for this in our first semester, and its motivation has been pretty well insured. Also, we have the chance to use our calculus processes in the study of this material; this use keeps them alive, and makes the geometric material itself more interesting.

These remarks emphasize that our main difficulty is encountered in getting our course *started*. Once we get it rolling, and once we have accumulated an adequate backlog of concepts and techniques, the rest of of the journey is comparatively smooth.

Our study thus far has been confined to a definite frame of reference - that of the pre-engineering student. This is because my own experience with the problem has been largely in that field. I see no reason, however, why all our findings would not apply to mathematics majors. On this point I note with much interest the experience of Reed College, where Professor Griffin teaches. In the May, 1951, issue of the American Mathematical Monthly, Professor Griffin reports that "The percentage of Reed male graduates since 1924 who have gone on to the doctorate in mathematics is the highest in the United States." He attributes this result in part to "the interest aroused by our introductory work in mathematics, which has led considerable numbers to select mathematics as their college major."

It is encouraging that during recent years there has developed a ferment of activity, largely exploratory, looking toward the creation of a course for non-mathematics majors. The suggested programs display extreme variety - in some cases even involving no prerequisite at all. Sometimes the selection of material seems to stem principally from the personal tastes of the writer; the "unit" idea quite generally prevails. This situation is, I suppose, inevitable, at least during the jelling period. But let me throw out one suggestion as to a possible selection criterion. We know that mathematics is "The language of Science." Suppose we ask, then, what core of mathematical material would best serve the student in his work in the natural and social sciences? Granting that specialists would need many years of mathematical work, what about the rank and file? Is there a fairly low common denominator? I have never seen evidence that anyone has seriously considered this problem. The connection with our present topic is this: if and when such a body of material has been agreed upon - should not the guiding principle for its presentation be the concept of unity? And will we not be inevitably drawn to J. W. Young's 1917 idea, *the concept of functionality?*

To summarize further: is it not true that the exuberant growth of all branches of knowledge now calls for *synthesis*? I believe that there are two main reasons why this is true. First, as a matter of efficiency and economy, we need to organize our materials so that we can reduce the time required to master the fundamentals. After all, there are vastly more things to learn than there were, even 50 years ago. This is true in the field of mathematics itself; but, equally importantly, the serviceability of mathematics in other fields of knowledge needs to be provided as rapidly and as early as possible. Second, we recognize the danger that details may obscure the pattern; that overall comprehension may not result from years of specialized courses. In the teaching of the sciences this question is now acute. The realization grows that technical competence may be achieved while no vision of the subject as a whole is developed; and that the general student may emerge from the early stages of this work with not even a feeling for the "scientific method of thought", I rejoice that the leaders in the teaching of science have begun to concern themselves with this problem. Their problem is more difficult than ours. For the inherent order and symmetry of mathematical modes of thought makes our search for unity more promising. I hope that we can take the lead in the endeavor, and that mathematics will set the pattern for the other fields, thus emphasizing her proud position as "Queen and Servant of the Sciences".

Los Angeles State College

AUTOMATIC COMPUTATION AS AN AID IN AERONAUTICAL ENGINEERING

John Lowe

The entire field of application for automatic computing machines is very large. In fact, if we admit machine accounting to this field, it affects all of us in a very personal way every day. However, my remarks are to be restricted to only the application of automatic computation in aeronautical engineering, and further restricted to consider only digital computers.

I shall first mention something of the reasons for automatic computation, its value and methods; discuss briefly its future, and lastly mention some of its difficulties.

First of all, what do we mean by automatic computation? As the experts would probably not agree exactly with my definition, let me beg the question and resort to example. The definition can be simplified by excluding analog computers, some of which are not truly computers at all but rather devices which simulate physical problems. By digital computer we mean a divice which solves problems digitally, or, literally, counts on its fingers. However from automatic computers we exclude desk calculators of the Friden or Marchant type. Specifically, then, this discussion is concerned with the use of IBM punched card equipment and such more sophisticated machines as the SWAC, the SEAC, BINAC, IBM's Selective Sequence Electronic Calculator, ENIAC, and certain similar machines presently being developed.

These latter machines are spectacular and have been popularized in newspapers and magazines. However, practical digital computing may be said to have started about eight years ago with the widespread use of IBM punched card equipment for this purpose. Even today, by far the largest share of the actual computing is being done with standard IBM equipment. These are essentially accounting machines; indeed, their proper name is Electric Accounting Machines. In the early days, we had a lot of trouble with such things as persuading IBM to change some machines to print minus signs instead of credit symbols to indicate negative numbers. The proper handling of algebraic signs presented considerable difficulty, and the first multiplier which provided a reasonably direct solution for this problem was regarded with awe, much as early automobile owners viewed the self-starter. However, today's machines have come a long way and are being improved at an ever increasing rate.

There must be some rather compelling reasons for the use of automatic computation. It is not cheap and provides plenty of headaches. Even a small-scale operation may cost $10,000 a month, and on a bigger scale we find that the cost of a single machine is in the order of magnitude of a million dollars. Hard-headed corporations do not spend sums like this without seeing a definite return.

A tremendous amount of numerical labor is involved in designing today's aircraft and missiles. These problems have increased enormously in the past ten years as the speeds of commercial aircraft have gone from 200 miles per hour to 350 miles per hour, and as the gross weight has gone from 20,000 pounds to 100,000 pounds. For example, in one phase of the fuselage stress analysis of a single configuration of the DC - 6 airplane, 200,000 multiplications and additions were performed, and the flutter analysis required 1,000,000 multiplications and additions. In these facts we find our first reason for the use of automatic computers. Mathematical labor of this magnitude can certainly be performed faster and more cheaply by automatic computing machines than by manual methods. Further, as the speed and complexity of aircraft increase, mathematical labor involved in their analysis becomes, for all practical purposes, impossible by manual methods and so we find that the machines are an indispensable feature of aircraft engineering.

It requires approximately 3 years and 1,000,000 man hours to engineer a large airplane from preliminary design to construction of prototype. This means that by the time a given airplane flies, it is obsolete in terms of the latest thinking. Obviously, any means which can be used to reduce this time lag is very valuable in terms of dollars and cents. Computing machines can be of material help in saving time, and, in fact, their ability to do things faster is probably of much more significance than their ability to do them cheaper.

Furthermore, the use of machines enables the engineer to make a much more thorough and detailed analysis of his designs than would be practical without them. This results in a better understanding of a given design and consequently a better finished product.

Another important factor is simplification. Much of an engineer's time is spent in devising simple methods of solution for particular problems so that they will be amenable to ordinary methods. When machines are used, it is often possible to ignore the simplifications and attack problems from first principles, using what is sometimes called the lead-pipe approach.

So we find that high-speed computation is encroaching on many phases of aeronautical engineering. Some of these are: Stress Analysis; Flutter Analysis; Harmonic Analysis; Wind Tunnel Data Reduction; Missile Trajectories; Stability Study; Catapult Launch Analysis; Supercharger Design; Airfoil Design; Auto-Pilot Design; Aeroelastic Analysis; Performance Calculations; Noise Level Studies; Heat Transfer in Jet Engines; Air Conditioning System Analysis; Ejectable Cockpit Trajectories; Weight Control; Data Reduction of Various Kinds; Radome Design. Of course, many of these problems fall into fixed mathematical patterns, such as the solution of simultaneous equations, matrix multiplication, solution of differential equations, the various techniques of data reduction, curve fitting, etc.

The future of machine computing depends on two factors: the development of new machines and the development of our ability to use them. In this respect, the aircraft industry is perhaps a bit peculiar. Many fields of application for computing machines have certain large specific problems requiring equipment of a particular capacity. We, in aeronautical engineering, on the other hand, have a variety of relatively small problems. Our past experience, which we expect to extend into the future, has been that as machines of greater and greater power are developed, so do we find greater and greater uses for them. This puts a challenge squarely up to the computing people to show engineers how machines can aid them.

The computing machine designers show every sign of performing nobly their share of this task. Of particular interest to us at Douglas is a machine being developed by IBM which we expect to use. This machine will be many times as powerful as any we now have. It will incorporate extremely high speed, large internal memory or storage, great flexibility and rapid input and output. We estimate that this machine will solve problems in 5 minutes which presently require 16 hours, and which would require 2,000 man hours if desk calculators were used.

The advent of such equipment presents a sharp challenge to computing personnel. Its potential value is enormous, but to realize this value will require the development of many new ideas and techniques.

What has been said may have left the impression that the advent of machines has by some Alladin's lamp magic solved all of our problems. This is not the case.

Many difficulties arise in the application of machines to specific problems. The first of these, perhaps, is the fact that these machines, like all other machines, are not infallible. Secondly, machines have no brains, and in saying this I must trust that I do not offend any of my good friends among the machine designers.

Machines must be programmed in the most minute detail. Problems which one does not consider in computing with a desk calculator become of paramount importance. For example, the determination of whether or not a given quantity is zero may require some planning. If data in graphical form are to be introduced into a computation, these data must be translated to numerical form, perhaps by some curve fitting method.

Programming can usually be broken into three phases: stating the problem, solving it, and checking the solution. Stating the problem is often the most difficult of these. All contingencies must be taken into account such as, perhaps, the proper procedure in case a number of which the machine is to obtain the square root should prove to be negative. It may be necessary to examine every number used anywhere in the calculation to be sure that it does not exceed the capacity of the machine on the one hand, or become so small as to lose necessary accuracy on the other.

Again, there is no royal road to success in the matter of checking solutions. This is often more difficult than solving the problem. Various methods are employed, the choice being dictated by circumstances. In the fortunate case, certain mathematical relationships exist among the results obtained and these can be checked. For example, if we are solving simultaneous equations, it is easy to check the solution mechanically by substituting the computed unknowns into the given equations. Differencing provides an excellent check on some problems, such as the solution of differential equations. In the worst cases, no such independent check can be devised and it becomes necessary to simply repeat the calculations.

Specialized mathematical techniques have been developed for high-speed computing machines. Their basis is, of course, numerical analysis. Their keynote is simplicity and flexibility. Much of the existing body of mathematics strives for labor saving methods, whereas for machine use we often find that these factors of simplicity and flexibility are definitely more important. For example, many methods exist for obtaining roots of polynomials, each one perhaps best under a given set of circumstances. However, in setting up a machine routine for solving polynomials, we are much more interested in a single formula which will be as widely applicable as possible, than we are in saving labor in a given instance.

Some examples of these methods are:

1) Any rational root of a number can be calculated using Newton's method as follows:

$$x = N^k$$

$$x_0 \simeq x$$

$$x_{n+1} = x\ [(1 - k) + \frac{kN}{x_n^{1/k}}]$$

If $k = ½$ $(x = \sqrt{N})$ then this becomes

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$$

These formulae are widely used in automatic machines.

2) The general arctangent function is messy to handle because it is discontinuous. Therefore, we avoid it when possible. For instance if we wish to find the value of $\sin(z + a)$, a is a constant, and $z = \tan^{-1}\frac{y}{x}$ we note that

$$\sin z = y(x^2 + y^2)^{-½},$$

$$\cos z = x(x^2 + y^2)^{-½},$$

and hence,

$$\sin(z + a) = \sin z \cos a + \cos z \sin a.$$

3) The numerical integration of differential equations has long been of interest and many methods have been developed for this purpose. Many, if not most, of them involve the use of prior points in extrapolating. For several reasons prior points are not convenient when automatic machines are used, so simpler methods are sought such as iteration, linear extrapolation, and the use of higher derivatives. Much remains to be learned about this subject.
4) One of the problems of handling large sets of numbers is keeping track of them and properly identifying them. Matrices can be of service in this connection as they provide a ready made coding system and logical algebra and calculus for manipulating arrays. Again, matrices are seldom of interest where manual methods are employed, but are proving to be extremely useful where machines are used.

To sum up, automatic computation is of great and growing importance in the aircraft industry. We are finding that with its aid we can design better aircraft faster and more cheaply. It also seems that its role will become an increasingly important one as aircraft become faster and more complex, and as engineering them becomes correspondingly more difficult. Further, we can be assured that new machines will be developed adequate to these greater problems. So far we have not mentioned the people required to operate these machines, but here lies a very significant segment of our problem.

The word "automatic" as applied to computing machines is relative. A Friden desk calculator is automatic; if two numbers are set on its keyboard and the proper button pressed, the product of the two numbers will be developed in a particular register. However, the machine conveys nothing about the meaning of the numbers involved; a meaning exists only in the mind of the operator. Further, the product of our example may or may not be correct. If the operator set up the factors correctly and pressed the proper operation button, and if the machine functioned properly, the correct product will be obtained. But this is not necessarily the end of the story. If the accuracy of one or both of the factors had been so vitiated by previous operations in which insufficient significant figures were retained, our product may be mathematically correct but physically meaningless.

Our "automatic" machines are only more automatic than desk machines. By pressing the proper button we can cause them to carry out long sequences of arithmetic operations. They can be programmed to use alternative sequences, the choice being based on the relative magnitudes of certain numbers developed in the course of the computation. Indeed, we shall shortly have machines which can handle sequences consisting of millions of steps. However, each step is like our desk calculator example. The numbers involved have meaning only in the mind of the operator and the final results will be meaningful only if both the operator and the machine have properly played their parts at every step.

It is not difficult to realize that this work demands a high degree of care, accuracy, clear-headedness and plain hard work. It also demands ingenuity to best adapt the tools available to the problem in hand. Infinite patience and forebearance are indispensible when the machines give trouble.

There exists a stringent shortage of people qualified for this work and this shortage shows every sign of becoming more acute. Capable people are being paid well. The field is so new that few people even know it exists. So little is known that ambitious people can be doing truly original work early in their careers. I hope that it will receive increasing recognition in school curricula and from student counselors. Today, few other fields offer technical or scientific college graduates the opportunity for advancement that is offered by computing.

Douglas Aircraft Co., Inc.

PROBLEMS AND QUESTIONS

Edited by

C. W. Trigg, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new and subject-matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by such information as will assist the editor. Crdinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in india ink and twice the size desired for reproduction. Readers are invited to offer heuristic discussions in addition to formal solutions.

Send all communications for this department to *C. W. Trigg, Los Angeles City College, 855 N. Vermont Ave., Los Angeles 29, California.*

PROPOSALS

105. *Proposed by J. S. Cromelin, Clearing Industrial District, Chicago.*

Mrs. Minnie Moscowitz left home with no money. She stopped at the bank to cash a check. Through an error which neither she nor the teller noticed, she was given as many dollars as her check read cents, and as many cents as the check read dollars. Next, Mrs. Moscowitz bought some stockings. She bought as many stockings as the check read dollars, and paid as much per stocking as the check read cents. She then counted her money and found she had just four times the amount of the check. How much was the teller short at the end of the day?

106. *Proposed by E. P. Starke, Rutgers University.*

However large n may be, show that there exist n consecutive integers each of which is divisible by a perfect square.

107. *Proposed by R. E. Horton, Lackland Air Force Base, Texas.*

Given circle C_1 intersecting circle C_2 in points R and T. From any point Q, lines QR and QT are drawn intersecting circle C_1 at A_1 and B_1, respectively, and circle C_2 at A_2 and B_2, respectively. Prove that A_1B_1 is parallel to A_2B_2.

108. *Proposed by H. S. M. Coxeter, University of Toronto.*

Consider a set of four equal spheres, of radius $\sqrt{2} + 1$, all touching one another, and another similar set, of radius $\sqrt{2} - 1$. Prove that the two sets can be so placed that each sphere of either set touches three of the other.

109. *Proposed by Leo Moser, Texas Technological College.*

In how many ways can an 8×8 chessboard be cut into two congruent pieces if the cuts are to be along the edges of the squares of the board?

110. *Proposed by H. T. R. Aude, Colgate University.*

a) Place a unit square with its sides parallel to the coordinate axes so that one curve of the family $x^2y = c$ will pass through two opposite corners and divide the area of the square in the ratio 1 : 3.
b) Consider the similar problem when the square is replaced by a rectangle a by b and the ratio of the division of its area is the proper fraction $n : m$.

111. *Proposed by P. D. Thomas, U. S. Coast and Geodetic Survey, Washington, D. C.*

A projectile is fired at an angle of elevation θ and with initial velocity u. After a time t_1 the projectile is at a point P where it suddenly receives an added velocity v directed along the tangent to the trajectory at P. Find an expression for the range of the projectile in terms of θ, u, t_1, and v. (Consider gravity as the only force acting.)

SOLUTIONS

Late Solutions

72. *Monte Dernham, San Francisco, California.*

Radius of Curvature of a Parabola

73. [Sept. 1950] *Proposed by W. B. Carver, Cornell University.*

If P_1P_2 is any arc of a parabola, show that the radius of curvature at some interior point of the arc is greater than the length of the chord P_1P_2.

Solution by Fred Marer, Los Angeles City College. We observe that for $0 \leqq y < ¼$, $4y < ¼ + 3y < ¼ + 3y + 12y^2 + 16y^3 = (1 + 4y)^3/4$. For $¼ \leqq y$, let $y = ¼ + \epsilon$, then $(1 + 4y)^3/4 = 2 + 12\epsilon + 24\epsilon^2 + 16\epsilon^3 > 1 + 4\epsilon = 4y$. Therefore, for all $y \geqq 0$,

$$4y < (1 + 4y)^3/4. \qquad (1)$$

Without loss of generality, consider the parabola, $x^2 = y$, on which lie the points $P_1(\pm\sqrt{y},\ y)$ and $P_2(\sqrt{y + \Delta y},\ y + \Delta y)$ where $y \geqq 0$, $\Delta y > 0$. Then

$$(P_1P_2)^2 = (\sqrt{y + \Delta y} \pm \sqrt{y})^2 + (y + \Delta y - y)^2 = 2y + \Delta y + (\Delta y)^2 \pm 2\sqrt{y^2 + y\Delta y}.$$

Hence

$$(P_1P_2)^2 < 2y + \Delta y + (\Delta y)^2 + 2\sqrt{y^2 + y\Delta y + (\Delta y)^2/4}$$
$$= 4y + 2\Delta y + (\Delta y)^2. \qquad (2)$$

The radius of curvature at any point (x, y) of the parabola is $R = (1 + 4y)^{3/2}/2$. It follows that R^2 is an increasing function of y for $y \geqq 0$ and will be greatest over the interval P_1P_2 at P_2 where it is

$$R_2{}^2 = [(1 + 4y) + 4\Delta y]^3/4$$
$$= (1 + 4y)^3/4 + 3(1 + 4y)^2\Delta y + 12(1 + 4y)(\Delta y)^2 + 16(\Delta y)^3. \qquad (3)$$

Now, by comparing like terms in the right-hand members of (2) and (3) and remembering (1), we have $(P_1P_2)^2 < R_2{}^2$ and $P_1P_2 < R_2$. But if $R_2 > P_1P_2$ at P_2, the continuity of $R = (1 + 4y)^{3/2}/2$ requires the existence of an interval overlapping P_1P_2 in which $R > P_1P_2$. Therefore, there exist interior points of the arc P_1P_2 at which the radius of curvature is greater than the length of the chord P_1P_2.

Also solved by *Charles McCracken, Jr., University of Cincinnati; and Samuel Skolnik, Los Angeles City College.*

A Game with a \$1.00 Bill

84. [Jan. 1951] *Proposed by B. F. Cron, Roxbury, Massachusetts.*

In a game which some of my friends play, one man holds a \$1.00 bill which has an eight-digit number imprinted twice on its face. Another man calls three digits. If these digits are in the imprinted number, he wins. For example, if *B* 27588607 *F* appears on the face of the bill and the second man calls 277, he wins. What is the probability of winning?

Solution by J. M. Howell, Los Angeles City College. It is assumed that the probability that any number from 00000000 to 99999999, inclusive, will appear on the face of the bill held by the first man is the same as the probability that any other number in the set will appear.

If the second man calls a number having three like digits, his probability of winning is

$$\sum_{i=3}^{8} \frac{8!(.9)^{8-i}(.1)^{i}}{i!(8-i)!} \text{ or } 0.03809179.$$

If the second man calls a three-digit number having two like digits, his probability of winning is

$$\sum_{i=1}^{6} \sum_{j=2}^{8-i} \frac{8!(.1)^{i}(.1)^{j}(.8)^{8-i-j}}{i!\,j!\,(8-i-j)!} \text{ or } 0.09197238.$$

If the second man calls a number having three unlike digits, his probability of winning is

$$\sum_{i=1}^{6}\sum_{j=1}^{7-i}\sum_{k=1}^{8-i-j}\frac{8!(.1)^i(.1)^j(.1)^k(.7)^{8-i-j-k}}{i!j!k!(8-i-j-k)!} \text{ or } 0.15426684.$$

Clearly the proper strategy for the caller is to call three unlike digits. The aspect of conscious choice by the caller may be removed by having him select his three-digit number from a Table of Random Numbers or by having him draw one slip from each of three boxes, each of which contains ten slips numbered with unlike digits. In this event his probability of winning is (0.01)(0.03809179) + (0.27)(0.0919728) + (0.72)(0.15426684) or 0.1139363083.

Two incorrect solutions were received.

Three Consecutive Integers

85. [Jan. 1951] *Proposed by Victor Thébault, Tennie, Sarthe, France.*

Find the three smallest consecutive integers each of which is the sum of two squares (zero excepted).

Solution by Monte Dernham, San Francisco. If m^2 is the largest square factor of an integer $N(= N_0 m^2)$, then N can be represented as the sum of two squares if, and only if, N_0 contains no prime factor of the form $4n - 1$. [See, e.g., Oystein Ore, *Number Theory and its History*, McGraw-Hill (1948), pp. 197, 267-271.]

Since one of every three consecutive integers is divisible by 3, a prime of the form $4n - 1$, it follows that one of the desired integers is divisible by 3^2. Now none of the first seven multiples of 9 belongs to a desired sequence. For 27, 54 and 63, N_0 contains a prime factor of the form $4n - 1$. Neither 9 nor 36 is representable as the sum of two non-zero squares. Both 18 and 45 are properly representable, but none of 16, 19, 44 and 46 are. Testing 72 next, we find that 71 is not, but 73 and 74 are both admissable. Hence the desired sequence is

$$72 = 6^2 + 6^2,\ 73 = 3^2 + 8^2,\ 74 = 5^2 + 7^2.$$

Continuing the investigation we find that the three smallest consecutive integers each of which is the sum of two *distinct* squares are

$$232 = 6^2 + 14^2,\ 233 = 8^2 + 13^2,\ 234 = 3^2 + 15^2.$$

Also solved by *Leo Moser, Texas Technological College; P. N. Nagara, College of Agriculture, Thailand; and the proposer.*

In other categories the smallest sets of integers each of which is the sum of two squares are:

Three consecutive even – $160 = 4^2 + 12^2$, $162 = 9^2 + 9^2$, $164 = 8^2 + 10^2$.
Two consecutive – $17 = 1^2 + 4^2$, $18 = 3^2 + 3^2$.
Two consecutive which are sums of *distinct squares* – $25 = 3^2 + 4^2$, $26 = 1^2 + 5^2$.
Two consecutive even – $8 = 2^2 + 2^2$, $10 = 1^2 + 3^2$.
Smallest representable in two ways – $50 = 1^2 + 7^2 = 5^2 + 5^2$.

Two Infinite Series

88. [Jan. 1951] *Proposed by O. E. Stanaitis, St. Olaf College, Minnesota.*

Establish the convergence or divergence of

a) $$1 - \frac{1}{2} + \frac{1}{3\sqrt{3}} - \frac{1}{4} + \frac{1}{5\sqrt{5}} - \frac{1}{8} + \frac{1}{7\sqrt{7}} - \frac{1}{16} + \cdots;$$

b) $$1 - \frac{1}{\sqrt{2}} + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} - \frac{1}{\sqrt{6}} + \frac{1}{7} - \frac{1}{2\sqrt{2}} + \cdots.$$

I. *Solution by M. S. Klamkin's Sophomore Calculus Class, Polytechnic Institute of Brooklyn, N.Y.* a) The given series may be written in the form $(1 + \frac{1}{3\sqrt{3}} + \frac{1}{5\sqrt{5}} + \cdots) - (\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots)$ thereby exhibiting it as the difference of two well-known convergent series. It follows that the given series is convergent.

b) Consider $\sum_{n=1}^{\infty} \left[\frac{1}{2n-1} - \frac{1}{\sqrt{2n}}\right] = \sum_{n=1}^{\infty} \frac{\sqrt{2n} - (2n-1)}{(2n-1)\sqrt{2n}}$. Since the degree of the denominator exceeds that of the numerator by only ½, the series diverges.

II. *Solution by L. A. Ringenberg, Eastern Illinois State College.* An easily proven comparison test not usually found in elementary calculus texts is: If Σa_n is a convergent [divergent] series of positive numbers and if $\lim_{n\to\infty} b_n/a_n = \pm 1$, then Σb_n is a convergent [divergent] series. This test is applied to the given series as follows:

a) Let $b_n = 1/(2n-1)^{3/2} - 1/2^n$ and $a_n = 1/(2n-1)^{3/2}$. Since Σa_n is convergent and $\lim_{n\to\infty} b_n/a_n = 1$, it follows that Σb_n is convergent.

b) Let $b_n = 1/(2n-1) - 1/\sqrt{2n}$ and $a_n = 1/\sqrt{2n}$. Since Σa_n is divergent and $\lim_{n\to\infty} b_n/a_n = -1$, it follows that Σb_n is divergent.

Also solved by *P. N. Nagara, College of Agriculture, Thailand; and the proposer.*

Regular Octahedron Inscribed in a Cube

94. [March 1951] *Proposed by Michael Goldberg, Washington, D. C.*

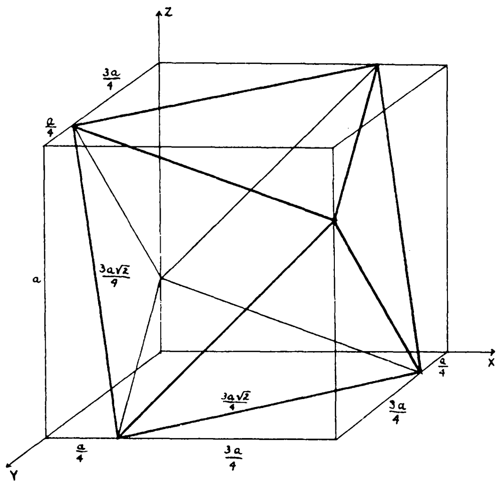
Find the largest regular octahedron which can be inscribed in a cube.

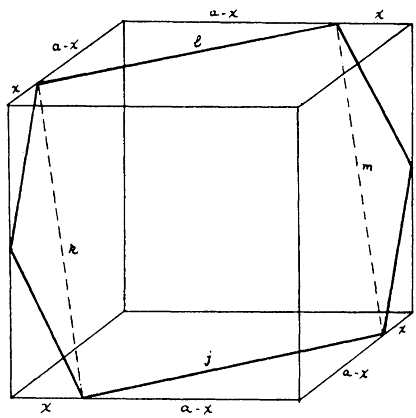
Solution by Leon Bankoff, D.D.S., Los Angeles, Calif. From each of two diametrically opposite vertices of the given cube, side a, mark off on the three concurrent edges segments equal to $3a/4$. Planes passed through each set of three extremities will be parallel to each other (since both are perpendicular to a diagonal of the cube) and both sections will be congruent, equilateral triangles (since the lines of intersection of the planes with the cube are hypotenuses of congruent isosceles right triangles). If lines are drawn joining each vertex of the two sections with the three vertices of the other, six of the lines complete the construction of an inscribed octahedron and the other lines are the three diagonals of the octahedron.

To prove that the inscribed octahedron is regular, it is sufficient to show that all twelve edges are equal. Each of the six edges of the original two sections is $3a\sqrt{2}/4$ by construction. Each of the remaining six edges is equal to $[(a/4)^2 + a^2 + (a/4)^2]^{1/2}$ or $3a\sqrt{2}/4$. This result can also be obtained analytically by letting three concurrent edges of the cube coincide with mutually perpendicular x, y, and z axes. The vertices of the octahedron will have the coordinates $(a, a, 3a/4)$, $(0, 3a/4, a)$, $(3a/4, 0, a)$, $(a, a/4, 0)$, $(a/4, a, 0)$, $(0, 0, a/4)$. By the familiar distance formula all of the edges are found to be equal to $3a\sqrt{2}/4$ or approximately $1.0606602\, a$.

To prove that this regular octahedron is the largest that can be inscribed in a cube, consider the three mutually perpendicular planes which pass through the twelve edges and the center of a regular octahedron. These planes cut the octahedron in congruent, square sections. In any possible position that an inscribed regular octahedron may occupy in a cube, the relationship of any one of the three square sections to the faces of the cube is symmetrically identical with that of the other two square sections to corresponding faces, for the cube is formed by the intersection of three parallel and equidistant pairs of mutually perpendicular planes. The problem, therefore, is reduced to finding the largest square that can be inscribed in a cube.

Planes can cut a cube in triangular, rectangular, pentagonal or hexagonal sections. Elementary considerations point to the conclusion that the maximum inscribed square will lie in a plane that intersects all six faces of the cube. In the hexagonal section thus formed, we are seeking two opposite, parallel sides, equal in length, the extremities of which will form a square when joined. To be equal in length, these lines must be parallel to and symmetrically equidistant from the diagonals of the faces of the cube in which they lie. Consequently the ratios of the line segments cut in the edges of the cube will be equal. Calling the shorter segment x, and the longer segment $a - x$, and letting j, k, l, m represent the sides of the square under consideration, we have $j^2 = (a - x)^2 + (a - x^2)$ and $k^2 = x^2 + a^2 + x^2$. Since it is desired that $k = j$, then $2(a - x)^2 = a^2 + 2x^2$ or $a^2 - 4ax = 0$. It follows that $x = a/4$ and $a - x = 3a/4$.





Similarly, the other vertices are definitely located on the edges of the cube, resulting in a regular octahedron of side $3a\sqrt{2}/4$. Since this is the only constructible inscribed regular octahedron with all vertices lying on the edges of the cube it is clearly a maximum.

Also solved by *the proposer*.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 42. The hypotenuse and one leg of a right triangle are 613105 and 613073, respectively. Find the other leg.

Q 43. State a theorem about integers which is valid for all integers n, with the exceptions $n = 5, 17, 257$. [*Submitted by Leo Moser.*]

Q 44. The midpoint of the join of the midpoints of the diagonals of a quadrilateral is the intersection of the joins of the midpoints of the sides. [*Submitted by B. K. Gold.*]

Q 45. Evaluate $\int_0^{\pi/2} \sin^2 x \, dx$ without use of the double-angle formulae. [*Submitted by Leo Moser.*]

Q 46. It is impossible to find integral solutions of $x^6 + y^6 + z^6 + x^2y + y^2z + z^2x + xyz = 0$. [*Emanuel Lasker in Scripta Mathematica*, 8, 14, (March 1941).]

Q 47. Given any ϵ with $0 < \epsilon < 1$, show that there exists a real number α such that the fractional part of α^n is greater than $1 - \epsilon$ for all positive integral n. [*Submitted by Leo Moser.*]

Q 48. Given the series obtained from two positive constants $u_1 = a < u_2 = b$, where $u_n = (u_{n-1} + u_{n-2})/2$, $n > 2$. Is this series convergent? [*Submitted by Fred Marer.*]

ANSWERS

A 42. $x = \sqrt{(613105)^2 - (613073)^2} = \sqrt{(613105 + 613073)(613105 - 613073)} = \sqrt{(1226178)(32)} = \sqrt{(841)(9^3)(64)} = (29)(27)(8) = 6264.$

A 43. $(n - 5)(n - 17)(n - 257) \neq 0.$

A 44. Let the quadrilateral be in the complex plane with vertices $A(z_1)$, $B(z_2)$, $C(z_3)$, $D(z_4)$. Let the midpoints of AB, BC, CD, DA, AC, BD be E, F, G, H, M, N, respectively. Then we have $E[\frac{1}{2}(z_1 + z_2)]$, $F[\frac{1}{2}(z_2 + z_3)]$, $G[\frac{1}{2}(z_3 + z_4)]$, $H[\frac{1}{2}(z_4 + z_1)]$, $M[\frac{1}{2}(z_1 + z_3)]$, $N[\frac{1}{2}(z_2 + z_4)]$. Hence $\frac{1}{4}(z_1 + z_2 + z_3 + z_4)$ is the midpoint of each of MN, EG, and FH.

A 45. Clearly $\int_0^{\pi/2} \sin^2 x\, dx = \int_0^{\pi/2} \cos^2 x\, dx = \frac{1}{2}\int_0^{\pi/2} (\sin^2 x + \cos^2 x) dx = \frac{1}{2}\int_0^{\pi/2} dx = \pi/4.$

A 46. It may be assumed that x, y, z have no common factor hence are not all even. But unless x, y, z are even the value of the left-hand member is always odd.

A 47. Let k be an integer such that $k - \sqrt{k^2 - 1} < \epsilon$. Let $\alpha = k + \sqrt{k^2 - 1}$ and $\beta = k - \sqrt{k^2 - 1}$. Clearly $\alpha^n + \beta^n$ is an integer for all n, and since $\beta^n < \epsilon^n$ the fractional part of α^n will be greater than $1 - \epsilon$.

A 48. *No.* $u_3 = (a + b)/2 > a$, and since the average of any two numbers greater than a is greater than a all the $u_i > a$, so the series diverges.

THE PERSONAL SIDE OF MATHEMATICS

This department desires especially articles showing what mathematics means to people in various professions and historical articles showing what classic mathematics meant to those who developed it. Material intended for this Department should be sent to the Mathematics Magazine, 14068 Van Nuys Blvd., Pacoima, California.

WHAT WE MEAN BY MATHEMATICS

Nilos Sakellariou

When we mention Mathematics, we, naturally, do not mean simple arithmetic which any moderately educated person must know, but what we really mean is Geometry and Analysis.

In History there were people, like the Aztecs who, without knowing Mathematics to a considerable extent, attained a high civilization and distinguished themselves in Art. It would be, therefore, possible to imagine a type of school where no Mathematics is included in its curriculum. The main reason why Mathematics is the center of gravity in modern schools all over the civilized world is, we believe, tradition and custom.

The Schools of Constantinople can be considered as posterity of the schools of the Hellenistic period (350 B.C. - 650 A.D.) where Mathematics was considered the main subject of learning. During that period lived Euclid, Aristarchus, Hipparchus, Eratosthenes and many others. But what is the main reason that Mathematics was considered a primary subject?

Aristotle defined Mathematics as the science of proportion and order. Everything treated by Mathematics is accomplished through logic and not by means of experiment, with the exception of a few elementary ideas used fundamentally and which we want to be in agreement with the outer world. The science of Mathematics is always in progress, without stopping or retreating, is never connected with fallacies or prejudices, does not depend on other sciences which owe their existence to the objective world or to a certain epoch, nor is it based on observation with the exception of a very few instances.

Descartes believed that the form of perfect truth was to be found in Mathematics and especially in Geometry. Mathematics is based on accurate definitions and axioms always reaching safe deductions due to the fact that logic is always kept in mind. That is why we often say that something is done with *mathematical accuracy* and that is why the Greeks, spoke of Mathematics as the *preeminent science* (κάτ' ἐξοχήν ἐπιστήμη).

Mathematics is by far superior to other sciences and more comprehensive. This was recognized by the Ancient Greek and Alexandrian civilizations as well as the philosophers and scientists of the latest centuries. It is based on a few original and basic ideas and is fully independent as against Physics, Mechanics, Shipbuilding, Ballistics, Aeronautics, Atomic energy, political, geographic and agricultural sciences which make use of Mathematics.

Finally the main reason for teaching Mathematics in schools is not for students to learn Mathematical formulas but to learn how to think mathematically i.e. with mathematical accuracy and precision.

Athens, Greece

ANNOUNCEMENT OF TENSOR SOCIETY (INTERNATIONAL)

President: Professor Akitsugu Kawaguchi, Faculty of Science, Hokkaido University, Sapporo, Japan.

Membership: open to all interested persons.

Dues: $2.00 a year.

Journal: Tensor (two issues per year).

Dues may be sent either by money order directly to Professor Kawaguchi or by check or other means to H. V. Craig, 3104 Grandview St., Austin, Texas.

MISCELLANEOUS NOTES

Edited by

Charles K. Robbins

Articles intended for this Department should be sent to Charles K. Robbins, Department of Mathematics, Purdue University, Lafayette, Indiana.

A SIMPLE GAME

On page 278 of the May-June issue (1951) of the Mathematics Magazine W. A. Nichols raises the question of the truth or falsity of the quotation from the Reader's Digest. A brief analysis from the Theory of Games will show the statement to be false.

Consider the coin tossing to be a game between the Caller and the Listener. Von Neuman (1) shows that such a game can be represented by a matrix of the form:

		Player 1 (Caller) Heads	Tails
Player 2 (Listener)	Heads	x	$-x$
	Tails	$-x$	x

where $\$x$ is the amount the Caller gains if he wins and $\$-x$ is the amount he may lose. Thus, the matrix shows that if the Caller choses Heads and the Listener tosses a Head, then the Caller wins $\$x$, etc.

Von Neuman defines a strategy for player 1 to be the probability with which he plays the various columns of the matrix and for player 2 the probability with which he plays the rows of the matrix. Let a_1, a_2 be the strategy for the Caller and b_1, b_2 be the strategy for the Listener. Then $a_1 + a_2 = 1$ and $b_1 + b_2 = 1$ and the $a_i \geq 0$, $b_j \geq 0$ as the probabilities of all the ways an event can happen are non-negative numbers whose sum is unity.

If the elements of a game matrix are x_{ij}, and player 1 has strategy a_i and player 2 has strategy b_j, then the value of the game (Ultimate outcome) is defined to be

$$v = \sum_{i,j} a_i b_j x_{ij}$$

That is, using these strategies, in the long run player 1 is guaranteed a gain of v. (Note that v may be negative)

Now in the particular game at hand the amount of the check is $\$2x$. Let the Caller use strategy $a_1 = .70$ and $a_2 = .30$. The strategy of the Listener is determined by the toss of the coin so $b_1 = b_2 = .5$. Thus the value of the game is

$$v = a_1 b_1 x - a_2 b_1 x - a_1 b_2 x + a_2 b_2 x$$

$$= (.7)(.5)x - (.3)(.5)x - (.7)(.5)x + (.3)(.5)x = 0.$$

This implies that in the long run by calling Heads 70% of the time and Tails 30%, the Caller will not lose any more than he gains. In fact it is easily seen that for any arbitrary strategy chosen by the Caller the value of the game is zero as

$$v = \tfrac{1}{2}a_1 x - \tfrac{1}{2}a_2 x - \tfrac{1}{2}a_1 x + \tfrac{1}{2}a_2 x \equiv 0$$

Therefore the statement from the Reader's Digest is false.

REFERENCES

(1) Von Neuman & Morgenstern, Theory of Games and Economic Behavior.
(2) Rand Corp., Zero Sum Games with a Finite Number or a Continuum of Strategies.

Lackland Air Force Base Lieut. Colonel Robert E. Horton

Comments on Note by Mathis in March-April Issue

The method given by Mathis, Mathematics Magazine Mar-Apr 1951 p. 224, for solving the impedance formula on the slide rule is ingenious and simple. Indeed, it is a shade easier than the one I have been using for about 40 years in that he uses two settings of the cross hair instead of two settings of the slide. My method is as follows:

1. Set 28 on the *C* scale against 18 on the *D* scale, using the cross hair. Note that this seems to be dividing 18 by 28, but by everting the scales it also divides 28 by 18, the result (1.555) appears on the *C* scale at the index on the *D* scale. It is not necessary to read this quotient.
2. On the *B* scale read the square of this quotient (2.42) at the index of the *A* scale.
3. Move the slide until 2.42 + 1, that is 3.42, on the *B* scale is under the index of the *A* scale, whereupon the result (33.3) appears under the cross hair on scale *C*.

It will be noted that this requires no more complicated rule than the Mannheim.

With either method, obtaining the angle is not at all difficult. In his step 2, Mathis has the tangent of the angle (1.555) on the *C* scale under the cross hair; or if the rule is at least a polyphase (not necessarily duplex), the cotangent on the *CI* scale. In my method, the last setting is $\frac{r}{z}$, the power factor, the cosine of the angle (on *D* scale under the index of the C scale).

It seems to me that Mathis's method is faster when Z only is needed. If the angle is needed also, I gain a bit due to the fact that the function appears at the last setting, whereas he must make a note of the tangent before proceeding. And finally, in those problems, not rare, where the power factor, the cosine, rather than the angle, is needed, my method gives it without further setting of either slide or cross hair.

I apologize if it is improper to call this method "mine". I have taught it to many, and have yet to see it used by one to whom I did not teach it.

H. K. Humphrey

Formula for Finding the Day of the Week, for Any Date in the Gregorian Calendar.

1. To the last two figures of the year add 1/4th of its value, disregarding any fractional remainder.
2. Add the Centurial Value - for 1500 add 5, for 1600 add 4, for 1700 add 2, and for 1800 add 0, for 1900 add 5, for 2000 add 4, for 2100 add 2, and for 2200 add 0.
3. Add the Month Value, according to the following table.

January	3	June	0
January Leap	2	July	2
February	6	August	5
February Leap	5	September	1
March	6	October	3
April	2	November	6
May	4	December	1

4. Add the date of the month.
5. Total the results of Year, plus its 1/4th, together with the centurial value, month value, and the date. Divide the above result by 7, and the REMAINDER will give the Day of the Week - the answer sought, according to the day values.

Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
1	2	3	4	5	6	0

Example: July 4th, 1951.

1/4th of 51	12	Disregard the remainder
Centurial value 1900	5	
July Value	2	
Date	4th	

Divide by 7)74 (10
70

Remainder 4, the value of 4 is Wednesday.

John Hoeck

Who devised this type of formula, and when was it discovered?

What is the scientific or technical term applied to this type of formula?

CURRENT PAPERS AND BOOKS

Edited by

H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to H. V. Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.

Basic Mathematical Analysis. By H. Glenn Ayre. McGraw-Hill Book Co., New York, 1950. $5.00

This text is a worthy contribution to the furthering of better mathematics at the very beginning of the college curriculum. The author's skillful handling of the development of the number system without alienating the student with too "cold" an axiomatic approach is in line with this trend.

The unifying theme of the book is the functional relation but unity is not forced at the expense of a thorough treatment of each topic considered. The range of the text, which is certainly designed for a year's work, consists of the conventional algebra, trigonometry and analytics with enough of calculus to give a deeper insight into polynomial functions.

Unusual features which deserve notice are the introduction of the number system in a logical fashion, the use of the slide rule, the theory and use of hyperbolic functions, and finally, the introduction of worthwhile historical notes at the end of each chapter.

At the end of the book answers are provided for all odd-numbered problems and a more adequate collection of tables than usual is supplied. These tables include: powers and roots, logarithms, trigonometric functions, natural logarithms, exponential and hyperbolic functions, and a radian conversion table.

The sketches in the book are well drawn with the exception of the one appearing on page 295; it leaves the unfortunate impression that corners may appear in the graph of a polynomial.

Lloyd L. Lassen

Elements of Ordinary Differential Equations. By Michael Golomb and Merrill Shanks. McGraw-Hill Book Co., New York, 1950. ix + 356 pages. $3.50.

This book embodies the subject matter which might be roughly designated as a first year's work in differential equations.

A good many noteworthy features have been incorporated into this text: an introductory chapter summarizing briefly the ideas and techniques drawn from algebra and calculus which will prove useful in later chapters; a careful presentation of existence theorems in the first two appendices; emphasis upon geometric and numerical approaches to the solution of problems in earlier chapters; a systematic and sound development of elementary operational calculus in later chapters; a precise treatment of solutions of classical equations such as Gauss' hypergeometric, Bessel's and Legendre's equations by power series; and even such unusual topics for a beginning text as the discussion of linear equations with periodic coefficients and an introduction to the problem of stability. All of these features combined with a splendid collection of exercises relating to the work of the physical scientist or engineer as well as those of purely mathematical interest should serve to make this an unusually fine text.

It should be mentioned that there is a short bibliography appended bearing directly upon the contents of the book. Furthermore, almost all the exercises have answers given in the back.

Lloyd L. Lassen

Die Entwicklungs-Geschichte Der Leibnizschen Mathematik. By Jos. E. Hofmann. Leibniz Verlag Munchen, 1949. 252 pages.

It is no longer a matter of argument that the calculus did not spring full blown from the mind of either Newton or Leibniz. This little book is devoted to the study and a very detailed study it is of the mathematical experiences Leibniz had during his stay in Paris from 1672 to 1676. This book is obviously the product of infinite pains; the bibliographic references (arranged in chronological order) occupy a full 24 pages or about 10 percent of the space.

It is this reviewer's opinion that this small treatise should become a part of the working library of anyone interested in the history of mathematics and particularly in the origins of infinitesimal analysis.

Lloyd L. Lassen

Mathematics at Work. Edited by William A. Gager, University of Florida. Published by W. W. Rankin, Mathematics Department, Duke University, Durham, North Carolina. 1949. 128 pages. $2.00.

This publication contains the high lights of the ninth annual Mathematics Institute held at Duke University, Durham, North Carolina. These Institutes were started by Professor W. W. Rankin in 1941 and have been continued each year since that time.

The main papers included consist of the actual lectures, given at the Institute, which have been arranged under four subtitles: Horizons, Application of Mathematics, Pure Mathematics, and Teaching of Mathematics. The material covered by these lectures should be of interest not only to those who attended the Institute but also to all teachers of mathematics from junior high school to college level. Special emphasis is given to the applications of mathematics to industry and business, thus giving the teacher a richer source of material from which he can answer the question, "How is this mathematics used?"

In addition to the main papers interesting material relative to the work of the Institute is included, such as, the reports of the study group leaders, list of participants, and various other items of interest to teachers of mathematics.

University of California, Los Angeles Clifford Bell

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